Please let me (Saul) know if any of the problems are unclear or have typos. There are no problems to turn in from this exercise sheet.

Exercise 6.1. Suppose that (X, d_X) is a metric space. Suppose that $I \subset \mathbb{R}$ is closed and connected. Suppose that $\alpha\colon I \to X$ is a geodesic. Prove that α is continuous and injective.

Exercise 6.2. [Harder.] Suppose that $n \geq 2$. Prove that the hyperbolicity constant for $\mathbb{H}^n$ is $\log(1+\sqrt{2})$. [Hints: It suffices to prove this in $\mathbb{H}^2$. I used the upper-half space model and the integral definition of length.]

Exercise 6.3. Suppose that (X, d_X) is a metric space with finite diameter D. Prove that X is $D/2$–hyperbolic.

Exercise 6.4. [Challenge.] Suppose that $F = F_2$ is the closed, connected, oriented surface of genus two. We take $G = \pi_1(F)$ and $S = \{a, b, c, d\}$. We use the following presentation:

$$G \cong \langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} \rangle$$

Prove that the Cayley graph $\Gamma(G, S)$ is δ–hyperbolic; your proof should give an explicit upper bound on δ.

Exercise 6.5. Suppose that $S = \{a, b\}$ and define G as follows:

$$G = BS(1, 2) = \langle a, b \mid aba^{-1} = b^2 \rangle$$

Prove that G, with respect to the generating set S, is not δ–hyperbolic for any δ. [Hint: Prove that the slimness constant for the quadrilateral $Q_n = Q(a^n, a^n b, ba^n b, ba^n)$ is proportional to n.]

Exercise 6.6. [Challenge.] Suppose that G is a group. Suppose that $H < G$ is a finite index subgroup. Prove that G is hyperbolic if and only if H is.

Exercise 6.7. Suppose that (X, d_X) is a metric space. Suppose that $k > 0$. Suppose that $\alpha\colon [0, L] \to X$ is a k–local geodesic. Prove that α is continuous. By means of an example, show that α need not be injective.

Exercise 6.8. Suppose that $k > 0$. Prove that a k–local geodesic in a tree is a geodesic.

Exercise 6.9. Suppose that (X, d_X) is a δ–hyperbolic metric space. Suppose that $k > 4\delta$. Suppose that $\alpha\colon [0, L] \to X$ is a k–local geodesic. We proved in Lecture 18 that α is in fact injective. As part of the proof, we assumed (for a contradiction) that $\alpha(0) = \alpha(L)$. We then defined $\ell \in [0, L]$ to be any point supremising $d_X(\alpha(0), \alpha(\ell'))$.

- Prove that such an ℓ exists.
- Prove that $k \leq \ell \leq L - k$.