Please let me (Saul) know if any of the problems are unclear or have typos. Please turn in Exercise 5.4 (via Moodle) by noon on Friday of week seven (2025-02-21).

In the following problems we take G a group, $S \subset G$ a finite generating set, $\Gamma_S = \Gamma(G, S)$ the resulting Cayley graph, and $B_S(1_G, n)$ the ball of radius n about 1_G in Γ_S.

Exercise 5.1. Suppose that $H < G$ is a finite index subgroup. (By Exercise 4.5.1, we have that H is also finitely generated.) Prove that G and H have the same number of ends.

Exercise 5.2. Here we use $B(n)$ to denote $B_S(1_G, n)$. We say that two geodesic rays α and β in Γ_S are *end-equivalent* if, for all n, we have that the infinite connected components of $\alpha - B(n)$ and $\beta - B(n)$ lie in the same connected component of $\Gamma_S - B(n)$. Prove that end-equivalence is an equivalence relation.

Exercise 5.3. Prove that, for every infinite connected component A of $\Gamma_S - B_S(1_G, n)$, there is a geodesic ray α in Γ_S that lies in A.

Exercise 5.4. Suppose that $G = F(a, b)$ is the free group generated by $S = \{a, b\}$. Suppose that $w \in G$ is a non-identity element. Prove that w, under its natural action, fixes exactly two ends of G.

Exercise 5.5. Suppose that $H < G$ is a subgroup of index $d < \infty$. Prove, for every $g \in G$, that H meets the ball $g \cdot B_S(1_G, d - 1)$.

Exercise 5.6. We again use $B(n)$ to denote $B_S(1_G, n)$. Define $C(n)$ to be the union of $B(n)$ with all finite connected components of $\Gamma_S - B(n)$. Suppose that $g \in G$ lies in $\Gamma_S - C(n)$ and has $|g|_S \geq 2n + 1$. Prove that $C(n)$ is disjoint from $g \cdot C(n)$.

Exercise 5.7. [Harder.] Prove that a finitely generated group has zero, one, two, or infinitely many ends.