Please let me (Saul) know if any of the problems are unclear or have typos. Please turn in Exercise 3.13 (via Moodle) by noon on Friday of week five (2025-02-07).

Exercise 3.1. Prove the *universal property for finite presentations*: Suppose that $G = \langle S \mid R \rangle$ is a finite presentation. Suppose that $\bar{\phi}\colon F(S) \to G$ is the induced homomorphism. Suppose that H is a group, suppose that $\psi\colon S \to H$ is any map, and suppose that R lies in the kernel of $\bar{\psi}$. Then there is a unique homomorphism $\Psi\colon G \to H$ so that $\bar{\psi} = \Psi \circ \bar{\phi}$.

Exercise 3.2. Given algorithms for the word problems of (G, S) and (H, T), provide an algorithm for the word problem of $(G \times H, S \sqcup T)$.

Exercise 3.3. Suppose that S and T are finite generating sets for the group G. Prove that the word problem for (G, S) is solvable if and only if the word problem for (G, T) is solvable.

Exercise 3.4. Suppose that Γ is a connected graph. Prove that the *edge metric* d_Γ (on $V(\Gamma)$) is indeed a metric.

Exercise 3.5. Suppose that $\Gamma_S = \Gamma(G, S)$ is the Cayley graph of a group. Prove that G acts (on the left) on Γ_S via isometries of the edge metric d_S. Using this, prove the following:

1. $d_S(g, h) = |g^{-1}h|_S$ for any $g, h \in G$.
2. $|g^{-1}|_S = |g|_S$ for any $g \in G$.
3. $|gh|_S \leq |g|_S + |h|_S$ for any $g, h \in G$.

Exercise 3.6. Suppose that $G = \mathbb{Z}^d$ and that S is the standard generating set. Prove that the word metric for (G, S) is the restriction (to $\mathbb{Z}^d$) of the L^1 metric on $\mathbb{R}^d$. That is

$$d_S(x, y) = \sum_{i=0}^{d-1} |x_i - y_i|$$

(Here x_i is the *exponent sum* of the i^{th} generator in x.)

Exercise 3.7. [Challenge.] Find an algorithm that, given $d \in \mathbb{N}$ and a finitely presented group $G = \langle S \mid R \rangle$, decides if there is a non-trivial homomorphism $\phi\colon G \to \mathrm{SL}_d(\mathbb{C})$.

Exercise 3.8.

1. Prove that any cyclic subgroup of $\mathbb{Z}^d$ is undistorted.
2. Prove that any cyclic subgroup of $F(S)$ is undistorted.

Exercise 3.9. Prove that the cyclic subgroup $\langle z \rangle$, generated by

$$z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is distorted in the Heisenberg group $H_3(\mathbb{Z})$.

Exercise 3.10. Prove that the subgroup $\langle b \rangle$ is distorted in

$$BS(1,2) = \langle a, b \mid aba^{-1} = 2 \rangle.$$

Exercise 3.11. Suppose that $G = \mathbb{Z}^2$ and that S is the standard generating set. Draw all short-lex spanning trees in the Cayley graph $\Gamma_S = \Gamma(G, S)$.

Exercise 3.12. Suppose that $G = BS(1, -1) = \langle a, b \mid aba^{-1} = b^{-1} \rangle$. Set $S = \{a, b\}$. Draw all short-lex spanning trees in the Cayley graph $\Gamma_S = \Gamma(G, S)$.

Exercise 3.13. Suppose that (G, S) is a finitely generated group. Suppose that $\gamma = \gamma_{G,S}$ is the induced growth function. Prove the following.

1. $\gamma(m + n) \leq \gamma(m)\gamma(n)$ for all $m, n \in \mathbb{N}$.
2. $\gamma(m) \leq (2|S| + 1)^m$ for all $m \in \mathbb{N}$.
3. G is finite if and only if there is some n so that $\gamma(n + 1) = \gamma(n)$.

Exercise 3.14. Let $D(d, n)$ be the volume of the n–ball in $\mathbb{Z}^d$ (with respect to the standard generating set). Prove the following.

1. Prove that, for any fixed d, there are positive real constants a and b so that

$$a \cdot n^d \leq D(d, n) \leq b \cdot n^d$$

   for all $n \in \mathbb{N}$.
2. $D(d + 1, n + 1) = D(d, n + 1) + D(d, n) + D(d + 1, n)$ for all $d, n \in \mathbb{N}$.
3. $D(d, n) = D(n, d)$ for all $d, n \in \mathbb{N}$.

For more information, see `https://oeis.org/A008288`.