Please let me (Saul) know if any of the problems are unclear or have typos. There are no problems to turn in from this exercise sheet.

Exercise 2.1. Give the missing details in the proof of the universal property of free groups. That is, suppose that u, v, and w lie in $F(S)$ and that s lies in $S \cup S^{-1}$ and prove the following.

- Suppose that $s * w$ is reduced. Prove $\bar{\phi}(s * w) = \bar{\phi}(s)\bar{\phi}(w)$.
- Prove $\bar{\phi}(w^{-1}) = \bar{\phi}(w)^{-1}$.
- Suppose that $u * v$ is reduced. Prove $\bar{\phi}(u * v) = \bar{\phi}(u)\bar{\phi}(v)$.

Exercise 2.2. Show that a bijection $\phi\colon S \to T$ induces an isomorphism $\bar{\phi}\colon F(S) \to F(T)$.

Exercise 2.3. Suppose that S is a subset of a group G.

- Prove that the subgroup generated by S equals the intersection of the subgroups containing S.
- Prove that the normal closure of S equals the intersection of the normal subgroups containing S.

Exercise 2.4. Prove that the elementary matrices (normally) generate $\mathrm{SL}_d(\mathbb{Z})$. Do they (normally) generate $\mathrm{GL}_d(\mathbb{Z})$?

Exercise 2.5. Prove that $H_3(\mathbb{Z})$ has rank two. (That is, $H_3(\mathbb{Z})$ is generated by a pair of elements, but is not cyclic.) Is $H_3(\mathbb{Z})$ normally generated by a single element?

Exercise 2.6. Prove that $\mathbb{Q}$ is not finitely generated.

Exercise 2.7. Prove that $F(a, b)$ is not normally generated by the element a.

Exercise 2.8. Prove the *universal property of group presentations*. That is: suppose that $G = \langle S \mid R \rangle$. Let $\bar{\phi}\colon F(S) \to G$ be the induced homomorphism. Suppose that H is a group and that $\psi\colon S \to H$ is a function. Suppose that R is contained in the kernel of the induced homomorphism $\bar{\psi}\colon F(S) \to H$. Then there exists a unique group homomorphism $\Psi\colon G \to H$ so that $\bar{\psi} = \Psi \circ \bar{\phi}$.

Exercise 2.9. Prove the following isomorphisms.

- $\mathbb{Z} \cong \langle a, b \mid b \rangle$
- $\mathbb{Z}/n\mathbb{Z} \cong \langle a \mid a^n \rangle$
- $\mathbb{Z}^2 \cong \langle a, b \mid [a, b] \rangle$
- $\mathbb{Z}^3 \cong \langle a, b, c \mid [a, b], [b, c], [c, a] \rangle$

- The fundamental group of the genus two surface and $\langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} \rangle$.

Exercise 2.10. Prove that each of the following is isomorphic to the trivial group.

1. $\langle a, b, c \mid aba^{-1} = b^2, bcb^{-1} = c^2, cac^{-1} = a^2 \rangle$
2. $\langle a, b \mid ab^3a^{-1} = b^2, ba^3b^{-1} = a^2 \rangle$

Exercise 2.11. Draw the Cayley graphs for the following groups and generating sets.

1. D_{2n} the dihedral group, generated by adjacent reflections.
2. D_∞ the infinite dihedral group, generated by adjacent reflections.
3. $\mathrm{SL}_2(\mathbb{Z})$ generated by the matrices

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

4. $\mathrm{SL}_2(\mathbb{Z})$ generated by the matrices

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

5. $\mathrm{SL}_2(\mathbb{Z})$ generated by the matrices A and R.
6. [Hard.] $BS(1, 2) = \langle a, b \mid aba^{-1} = b^2 \rangle$ generated by a and b.

Exercise 2.12. Suppose that Γ is a connected non-empty graph. Prove that the following are equivalent.

1. Every (open) edge of Γ separates.
2. There is no embedded cycle in Γ.
3. Any two vertices in Γ are connected by a unique edge path.