Please let me (Saul) know if any of the problems are unclear or have typos. Please turn in Exercise 1.7 (via Moodle) by noon on Friday of week three (2025-01-24).

Exercise 1.1. Let $H_3(\mathbb{Z})$ be the set of three-by-three upper triangular integer matrices with all diagonal entries equal to one. Show that $H_3(\mathbb{Z})$, with the usual matrix multiplication, is a group. Give explicit formulas, in the given coordinates, for inversion and multiplication.

Exercise 1.2. Define $\mathrm{NIL} = \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ to be the semi-direct product where the group $\mathbb{Z}$ acts on the group $\mathbb{Z}^2$ via

$$\phi_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

Prove that NIL is isomorphic to $H_3(\mathbb{Z})$ by giving an explicit isomorphism.

Exercise 1.3. Suppose that U_n is the group of n^{th} roots of unity. Suppose that D_{2n} is the dihedral group of order $2n$. Prove that D_{2n} is isomorphic to $U_n \rtimes U_2$ where the latter acts on the former by inversion.

Exercise 1.4. Find all actions of $\mathbb{Z}$ on the following:

- the metric space $\mathbb{R}$, where $d_{\mathbb{R}}(x, y) = |x - y|$
- the vector space $\mathbb{R}$ (over the field $\mathbb{R}$)
- the (connected) graph $\mathbb{R}$, having countably many vertices all of of degree two.

(Morally, this is the same as computing the automorphism group in each case.)

Exercise 1.5. Let $S = \{0, 1\}$. So the elements of S^* are *binary strings*. Count the following:

- the binary strings of length n
- the binary strings of length at most n
- the binary strings of length n without 11 as a subword
- [Harder.] the binary strings of length n without 111 as a subword

Exercise 1.6. Prove the *periodicity lemma*: Suppose that S is a set. Suppose that u and v are words over S with $u * v = v * u$. Then there is a word w over S, and integers p and q, so that $u = w^p$ and $v = w^q$.

Exercise 1.7. Prove the *cancellation lemma* (for the free groups): Suppose that S is a set. Suppose that u and v are elements of the free group $F(S)$, thus are reduced words. Prove that there are elements u', v', and w of $F(S)$ so that

$$u = u' * w \qquad v = w^{-1} * v' \qquad u \cdot v = u' * v'$$

(It follows that u', v', and w are the unique such.)

Exercise 1.8. Suppose that S is a set.

- Prove that the free group $F(S)$ is commutative if and only if $|S| = 1$.
- Prove that $F(S)$ has no non-trivial torsion.
- [Challenge.] Prove that all subgroups of $F(S)$ are isomorphic to free groups. (Start with the finitely generated subgroups.)

Exercise 1.9. Let $\mathbb{Z}^m$ be the *free commutative group of rank* m.

- Prove that $\mathbb{Z}^m$ has no non-trivial torsion.
- Prove that $\mathbb{Z}^m \cong \mathbb{Z}^n$ if and only if $m = n$. (This justifies the name "rank".)
- Prove that $\mathbb{Z}^m$ surjects $\mathbb{Z}^n$ if and only if $m \geq n$.
- Prove that all subgroups of $\mathbb{Z}^m$ are finitely generated.
- Prove that all subgroups of $\mathbb{Z}^m$ are isomorphic to free commutative groups and have rank at most m.