2025-02-19 SAULSCHLEIMER LECTURE 20 MA4H4

OPEN SUPPOSE G HYPERBOLIC. IS THERE $H < G$ FIN INDEX WITH $H \neq G$?
WITH H TORSION-FREE?
IS G RESIDUALLY FINITE?

① FIN. PRESENTED

THEOREM: SUPPOSE (G,S) IS δ-HYPERBOLIC. THEN G IS FINITELY PRESENTED.

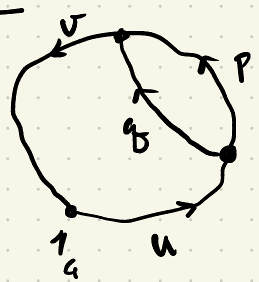
PROOF: DEFINE $R = \{ w \in F(S) \mid |w| \leq 8\delta + 1 \text{ AND } w =_G 1_G \}$.
SO $\text{CARD}(R) \leq (2|S|)^{8\delta+1}$ AND SO R IS FINITE.

CLAIM: $\langle\langle R \rangle\rangle = \text{KER}(F(S) \to G)$.

THIS IMPLIES $G \cong \langle S \mid R \rangle$ AND PROVES THE THEOREM.

PROOF OF CLAIM: SUPPOSE $w \in \text{KER}(F(S) \to G)$. IF $w \in R$ WE ARE DONE. SO SUPPOSE $|w| \geq 8\delta + 2$. SINCE $w =_G 1_G$ IT IS THE LABEL OF A LOOP IN $\Gamma(G,S)$. SO w IS NOT A $(4\delta+1)$-LOCAL GEODESIC. SO THERE IS A FACTORISATION $w = upv$ AND A GEODESIC WORD q SO THAT $q =_G p$, AND $|q| < |p| \leq 4\delta + 1$.

PICTURE:



SO: $p \cdot q^{-1} \in R$

AND: $u\, p \cdot q^{-1} u^{-1} \in \langle\langle R \rangle\rangle$

BY INDUCTION: $u q v \in \langle\langle R \rangle\rangle$

THUS

$$u p q^{-1} u^{-1} \cdot u q v =_S u p v = w \in \langle\langle R \rangle\rangle \quad \square$$

② DEHN PRESENTATIONS

DEF: A FINITE PRESENTATION $G = \langle S \mid R \rangle$ IS A DEHN PRESENTATION IF FOR EVERY $w \in \langle\langle R \rangle\rangle$ THERE IS SOME uv^{-1} SO THAT:

(i) $|v| < |u|$

(ii) u IS A SUBWORD of w

(iii) SOME ROTATION of uv^{-1} OR vu^{-1} LIES IN R.

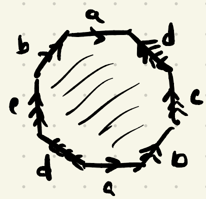
DEF: IF $w = u * v$ THEN $w' = v * u$ IS A ROTATION of w.

EXAMPLE: TAKE $F =$  AND

$$\pi_1(F) \cong \langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} \rangle$$

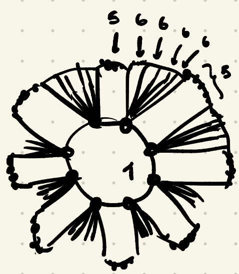
LEMMA [DEHN, 1912] $\pi_1(F)$ IS A DEHN PRESENTATION.

PROOF: WE REALISE F AS A QUOTIENT:



THE UNIV. COVER $\tilde{F}$ IS A PLANE TILED BY OCTAGONS, EIGHT ABOUT A VERTEX.

CARTOON:

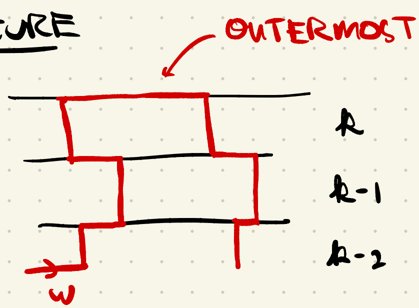


BETTER: SET D_0 = CENTRE TILE. D_{n+1} = UNION of ALL TILES WITH CLOSURE MEETING D_n. $S_n = \partial D_n$. $R_{n+1} = D_{n+1} - D_n$. COUNT TILES IN D_n, R_n. COUNT EDGES IN S_n. COUNT EDGES IN INTERIOR of R_n.

$\Gamma(G,S)$ IS THE ONE-SKELETON of THIS TILING. SUPPOSE w IS (THE LABEL of) A LOOP IN Γ. WE ASSUME w STARTS/ENDS AT 1_G. WE FREELY REDUCE w.

IF $w \neq \varepsilon_S$ THEN IT REACHES SOME OUTERMOST CIRCLE, TRAVELS ALONG IT SOME DISTANCE, AND THEN DESCENDS.

PICTURE



THE TILES IN THE k^{TH} RING MEET THE OUTER CIRCLE IN SUBWORDS of LENGTH 5 OR 6. THIS IS GREATER THAN 4, SO WE ARE DONE □