2025-02-13 LECTURE 18 SAUL SCHLEIMER MA4H4

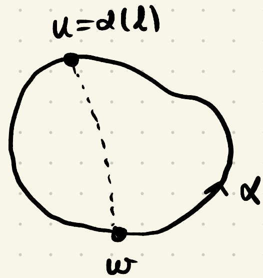
① LOCAL GEODESICS IN HYP SPACES

NON-TRIVIAL GEODESICS NEVER FORM LOOPS. THIS INSPIRES:

PROPOSITION: SUPPOSE X IS A δ-HYP. METRIC SPACE. SUPPOSE $k > 4\delta$ AND $L > 0$ [BOTH STRICT]. SUPPOSE $\alpha : [0,L] \longrightarrow X$ IS A k-LOCAL GEODESIC. THEN $\alpha(L) \neq \alpha(0)$

COROLLARY: WITH X, δ, k, α AS ABOVE: α IS AN EMBEDDING.

PROOF of PROP: SUPPOSE, FOR A CONTRADICTION, THAT $\alpha(0) = \alpha(L)$. SET $w = \alpha(0) = \alpha(L)$. PICK $\ell \in [0,L]$ THAT MAXIMISES $d_X(w, \alpha(\ell)) = d_X(\alpha(0), \alpha(\ell))$. PICTURE



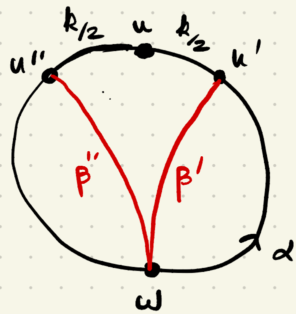
SET $u = \alpha(\ell)$.

EXERCISE: SINCE $d_X(w, \alpha(\ell))$ IS MAXIMISED AT $u = \alpha(\ell)$ WE HAVE

$\ell \geq k$ AND $\ell \leq L - k$ [SO $L \geq 2k$]

SET: $\ell' = \ell - k/2$, $u' = \alpha(\ell')$
$\ell'' = \ell + k/2$, $u'' = \alpha(\ell'')$

PICTURE



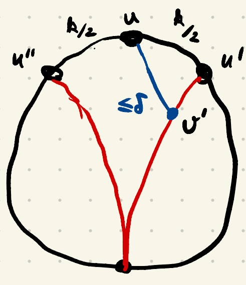
NOTE $\alpha|[\ell', \ell'']$ IS A GEODESIC.

PICK GEODESICS β', β'' FROM w TO u', u''.

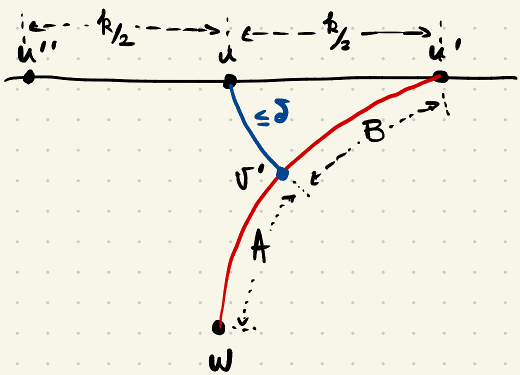
SO $(\alpha|[\ell', \ell''], \beta', \beta'')$ IS A GEODESIC TRIANGLE, THUS SLIM.

SUPPOSE (AS THE OTHER CASE IS SIMILAR) THERE IS SOME $v' \in \beta'$ WITH $d_X(u, v') \leq \delta$

PICTURE:



PICTURE:



SET $A = d_X(w, v')$

$B = d_X(v', u')$.

RECALL: $d_X(w,u') \le d_X(w,u)$

SO $A + B \le A + \delta$

SO $B \le \delta$

SO $k/2 \le 2\delta$

SO $k \le 4\delta$, CONTRADICTION. $\square$

[IDEA: $[u', u'']$ "BOWS OUT" BY DEF of u, BUT "BOWS IN" BY HYP. THIS GIVES THE CONTRADICTION...]

② NOT PURELY TORSION.

THEOREM: SUPPOSE G IS A GROUP, $S \subset G$ A FIN. GEN SET. SUPPOSE $\Gamma = \Gamma(G,S)$ IS δ-HYPERBOLIC. SUPPOSE DIAM$(\Gamma) \geq k \cdot (2|S|)^k + k$ FOR SOME $k > 4\delta$ [STRICT]. THEN G CONTAINS A NON-TORSION ELEMENT. [THUS DIAM$(\Gamma) = \infty$].

PROOF: BY THE ASSUMPTION ON DIAMETER, Γ CONTAINS A GEODESIC ARC α of LENGTH $k \cdot (2|S|)^k + k$. LET w BE THE LABEL of α. SO $w = v u v' u v''$ FOR SOME GEODESIC WORDS v, v', v'', AND u, WITH $|u| = k$. LET $g = uv'$. WE DEFINE PATHS γ_n WITH LABEL $g^n = uv'uv'uv' \cdots uv'$.

CLAIM: FOR ALL n THE PATH γ_n IS A k-LOCAL GEODESIC.

PROOF: ANY SUBPATH (of LENGTH k) HAS LABEL IN uv' OR IN $v'u$. SO IT HAS THE SAME LABEL AS A SUBPATH of w. SO THE SUBPATH IS A GEODESIC. $\square$

BY PROPOSITION γ_n IS NOT A LOOP, SO g IS NOT TORSION.

$\square$

EXERCISE: LOWER THE DIAMETER BOUND FROM $k\cdot(2|S|)^k + k$ TO $(2|S|)^k + k - 1$

CHALLENGE: GIVE ~~LOWER BOUNDS FOR THE DIAMETER BOUND, BY GIVING FINITE GROUPS (G,S) WITH $|G|$ LARGE(ISH) BUT $|S|$ AND δ SMALL(ISH).~~ [OR PERHAPS $\delta \approx \mathrm{DIAM}(\Gamma)$, VIA LARGE GIRTH, IS THE BEST WE CAN HOPE FOR? EG $PSL(2,p)$]

3 SIZE of TORSION ELEMENTS

BEFORE CONTROLLING TORSION SUBGROUPS WE GIVE AN EASIER BOUND:

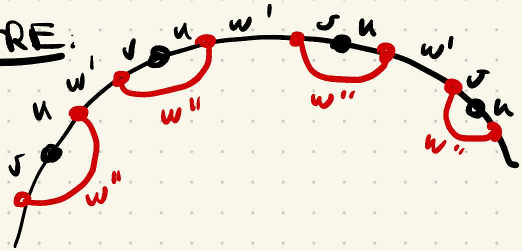
THEOREM: SUPPOSE (G,S) IS δ-HYPERBOLIC. SUPPOSE $g \in G$ IS TORSION. THEN THERE IS SOME $h \in G$ SO THAT $|hgh^{-1}|_S \leq 4\delta$.

COROLLARY: UP TO CONJUGATION, G HAS ONLY FINITELY MANY TORSION ELEMENTS. □

PROOF of THM: PICK ANY $h \in G$ SO THAT

$$|hgh^{-1}|_S = \mathrm{MIN}\{ |fgf^{-1}|_S : f \in G \}.$$ REPLACE g BY hgh^{-1}.

FOR A CONTRADICTION SUPPOSE $|g| \geq 4\delta + 1 = k$. SET $n = \mathrm{ORD}(g) < \infty$. SO $g^n = 1_G$. LET $w \in F(S)$ BE A WORD of LENGTH $|g|_S$ SO $w =_G g$. SO w^n IS A LOOP. SO w^n IS NOT A k-LOCAL GEODESIC (PROPOSITION). SINCE w IS GEODESIC AND $|w| = |g|_S \geq k$ THERE IS A FACTORISATION $w = uw'v$ WITH vu NOT GEODESIC AND $|vu| \leq k$. **PICTURE**:



LET $w'' \in F(S)$ BE A GEODESIC WORD WITH $w'' =_G vu$ AND

$|w''| < |vu|$

<u>SO</u>: $w'w'' =_G w'vu$ AND $w'vu =_G u^{-1}gu$

BUT $|w'w''| < |g|_S$. A CONTRADICTION. $\square$