2025-02-12 LECTURE 17 SAUL SCHLEIMER MA4H4

① HYPERBOLIC GROUPS:

DEF: SUPPOSE G IS A GROUP. WE SAY G IS A HYPERBOLIC GROUP IF THERE IS SOME FINITE GEN SET $S \subset G$ AND SOME $\delta \geq 0$ SO THAT $\Gamma(G,S)$ IS δ-HYPERBOLIC.

[CHALLENGE: ESTIMATE δ IN $\mathbb{Z}/2^n\mathbb{Z}$, $PSL(2,p)$, ...]

EXAMPLES

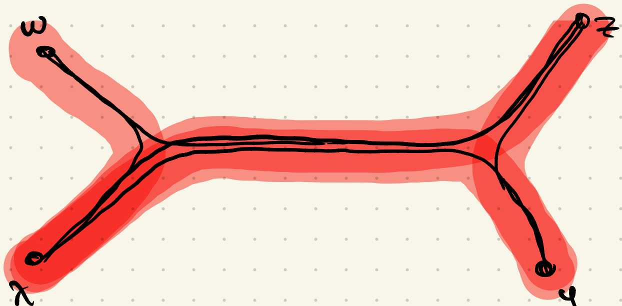
(1) FINITE GROUPS (ANY GEN SET)

(2) FREE GROUPS (STANDARD GENS)

③ $\pi_1(S_2) = \langle a,b,c,d \mid abcd\bar{a}\bar{b}\bar{c}\bar{d} \rangle$

LEMMA: IN HYP SPACE QUADRILATERALS ARE 2δ-SLIM

PROOF:



SUPPOSE Q IS A QUAD WITH VERTICES x,y,z,w AND GEODESICS $[x,y]$, $[y,z]$, $[z,w]$, $[w,x]$.

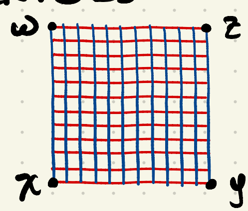
THEN THE 2δ-NEIGHBOURHOODS of $[x,y]$, $[y,z]$ CONTAIN THE δ-NEIGH of $[x,w]$. NOW CONSIDER THE TRIANGLE WITH SIDES $[x,z]$, $(x,w]$, AND $[w,z]$. □

NON-EXAMPLE: $\mathbb{Z}^2$ IS NOT δ-HYPERBOLIC FOR ANY δ.

PROOF: CONSIDER THE QUAD WITH VERTICES

$x = 1_G$, $y = a^n$, $z = a^n b^n$, $w = b^n$

THIS IS $n/2$-SLIM, BUT NOT $n/4$-SLIM. □



NOTE THAT THE DISK BOUNDED BY THE QUAD HAS "POOR ISOPERIMETRIC RATIO": ITS AREA IS APPROXIMATELY THE SQUARE of ITS PERIMETER.

EXERCISE: $BS(1,2) = \langle a, b \mid aba^{-1} = b^2 \rangle$ IS NOT HYPERBOLIC.

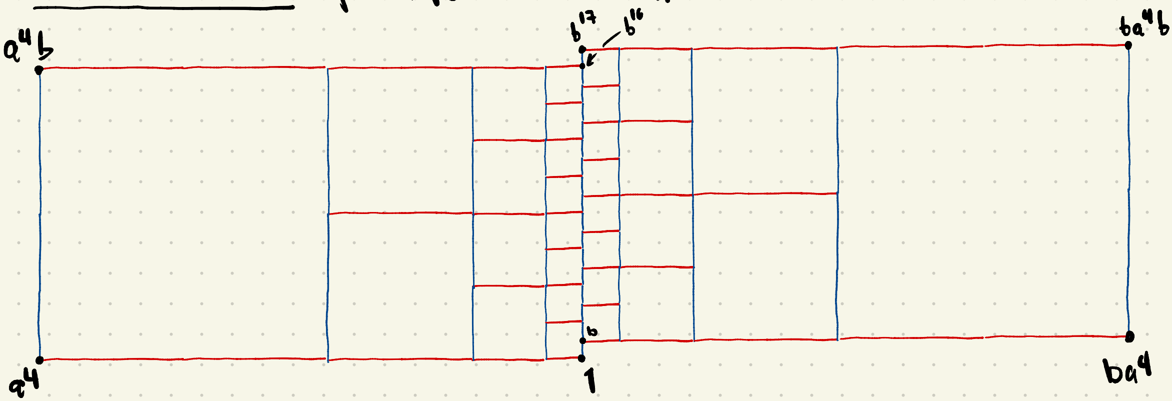
CONSIDER THE QUADRILAT. WITH VERTICES AT

$$x = a^n, \quad x' = a^n b, \quad y = ba^n, \quad y' = ba^n b.$$

HINT: CONSIDER AS WELL THE PTS $1_G, b, b^{2^n}, b^{2^n+1}$

THIS TIME THE AREA of THE QUAD IS EXPONENTIAL IN ITS PERIMETER!

BETTER VERSION of FIGURE DRAWN IN LECTURE



CHALLENGE: SUPPOSE G FIN GEN AND $H < G$ FIN. INDEX. THEN G IS HYPERBOLIC IF AND ONLY IF H IS.

---

(2) LOCAL GEODESICS: SUPPOSE X IS A GEODESIC METRIC SPACE. SUPPOSE $I \subset \mathbb{R}$ IS CLOSED, CONNECTED. A FUNCTION $\alpha : I \longrightarrow X$ IS A k-LOCAL GEODESIC IF, FOR ALL $a, b \in I$ WITH $|b - a| \leq k$ WE HAVE $\alpha|[a,b]$ IS A GEODESIC.

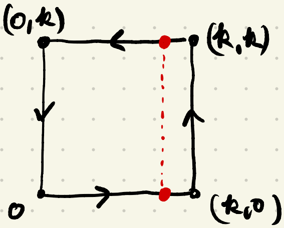
EXAMPLE: A GEODESIC IS A k-LOCAL GEODESIC FOR ALL $k \geq 0$.

EXERCISE: IN A TREE k-LOCAL GEODESICS (FOR $k>0$) ARE GEODESICS. [THIS IS ONE REASON TO PARAMETERISE BY CONTINUOUS INTERVALS INSTEAD OF DISCRETE ONES.]

EXAMPLE: FIX $k \in \mathbb{N}$. DEFINE $\alpha_k : [0,4k] \to \Gamma(\mathbb{Z}^2)$ BY

$$\alpha_k(t) = \begin{cases} (t,0), & \text{IF } t \in [0,k] \\ (k, t-k), & \text{IF } t \in [k,2k] \\ (3k-t, k), & \text{IF } t \in [2k,3k] \\ (0, 4k-t), & \text{IF } t \in [3k,4k] \end{cases}$$

PICTURE



THIS IS A k-LOCAL GEODESIC **LOOP** IN $\Gamma(\mathbb{Z}^2)$. [NOTE α_k IS NOT A $(k+2)$-LOCAL GEODESIC]

③ LOCAL GEODESICS IN HYP SPACES

NOTE THAT NON-TRIVIAL GEODESICS NEVER FORM LOOPS.

PROPOSITION: SUPPOSE X IS A δ-HYP. METRIC SPACE. SUPPOSE $k > 4\delta$ AND $L > 0$ [BOTH STRICT]. SUPPOSE $\alpha : [0,L] \longrightarrow X$ IS A k-LOCAL GEODESIC. THEN $\alpha(L) \neq \alpha(0)$.

REMARK: THUS α IS NOT A LOOP. AS A COROLLARY, k-LOCAL GEODESICS (WITH $k > 4\delta$) ARE EMBEDDINGS.

MORALLY: LOCAL GEODESICS IN HYPERBOLIC SPACES ARE ALMOST "STRAIGHT". WE WILL RETURN TO THIS.