2025-02-11 LECTURE 16 SAUL SCHLEIMER MA4H4

① METRIC SPACES

SUPPOSE (X, d_X) IS A METRIC SPACE

DEF: SUPPOSE $B \subset X$, $x \in X$. WE DEFINE

$$d_X(B, x) = \inf\{d(b, x) \mid b \in B\}.$$

WE DEFINE

$$N_X(B, R) = \{x \in X \mid d_X(B, x) \le R\}.$$

THIS IS THE R-NEIGHBOURHOOD of B IN X.

DEF: SUPPOSE $I \subset \mathbb{R}$ IS CLOSED, CONNECTED. [SO I IS AN ARC, RAY, OR LINE] SUPPOSE $\alpha : I \to X$ IS A FUNCTION. WE CALL α A GEODESIC IF

$$d_X(\alpha(a), \alpha(b)) = |b - a| \quad \text{FOR ALL } a, b \in I.$$

EXERCISE: GEODESICS $\alpha : [a, b] \longrightarrow X$ ARE ALWAYS CONTINUOUS AND INJECTIVE. [HOMEOMORP. TO IMAGE]

COROLLARY: NON-TRIV. GEODESICS ARE NEVER LOOPS. $\square$

DEF: A METRIC SPACE IS GEODESIC IF FOR ALL $x, y \in X$ THERE IS A GEODESIC α IN X FROM x TO y.

EXAMPLES:

(1) $\mathbb{E}^n$ EUCLIDEAN SPACE $[\mathbb{R}^n$ WITH L^2 METRIC$]$

(2) $\mathbb{H}^n$ HYPERBOLIC SPACE [SAY: OPEN UNIT BALL IN $\mathbb{R}^n$ WITH $\cosh(d_{\mathbb{H}}(x, y)) = 1 + \dfrac{2\|x - y\|^2}{(1 - \|x\|^2)(1 - \|y\|^2)}$, EASY!]

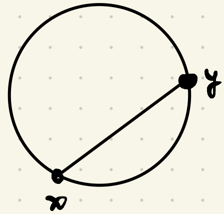
(3) S^n SPHERE [WITH ANGLE METRIC] [EXERCISE]

(4) Γ GRAPH WITH EDGE METRIC (EXTENDED TO EDGES)

NONEXAMPLES

(1) DISCRETE METRIC SPACE.

(2) S^n WITH CHORDAL METRIC

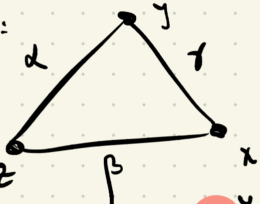


(3) $\mathbb{R}^2 - \{0\}$ WITH INDUCED METRIC

② HYPERBOLIC SPACES

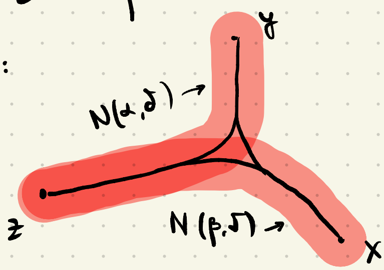
DEF: SUPPOSE $x, y, z \in X$. SUPPOSE $\alpha, \beta, \gamma \subset X$ ARE GEODESICS FROM y TO z, z TO x, x TO y. WE CALL THE TRIPLE $T = (\alpha, \beta, \gamma)$ A GEODESIC TRIANGLE IN X WITH VERTICES AT (x, y, z).

PICTURE:



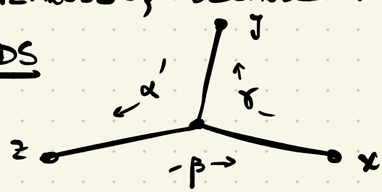
DEF A GEODESIC TRIANGLE $T = (\alpha, \beta, \gamma)$ IS δ-SLIM IF $\gamma \subset N_X(\alpha, \delta) \cup N_X(\beta, \delta)$.

PICTURE:



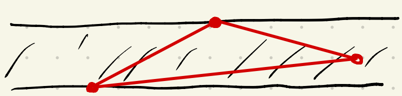
DEF: A METRIC SPACE IS δ-HYPERBOLIC IF ALL GEOD. TRIANGLES ARE δ-SLIM.

EXAMPLES: A TREE IS ZERO-HYPERBOLIC, BECAUSE ALL GEODESIC TRIANGLES ARE TRIPODS



EXERCISE $\mathbb{H}^n$ $(n \geq 2)$ IS $\log(1+\sqrt{2})$-HYPERBOLIC.

EXAMPLE: $\mathbb{R} \times [0,1] \subset \mathbb{R}^2$ WITH USUAL METRIC IS 1-HYPERBOLIC.



EXERCISE: SUPPOSE (X, d_X) IS A GEODESIC SPACE WITH $\mathrm{DIAM}(X) \leq \delta$. THEN X IS $\frac{\delta}{2}$-HYPERBOLIC. [EXAMPLE of C_n = CYCLE

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③ HYPERBOLIC GROUPS:

DEF: SUPPOSE G IS A GROUP. WE SAY G IS A **HYPERBOLIC GROUP** IF THERE IS SOME FINITE GEN SET $S \subset G$ AND SOME $\delta \geq 0$ SO THAT $\Gamma(G, S)$ IS δ-HYPERBOLIC.

EXAMPLES

(1) FINITE GROUPS (ANY GEN SET)

[**CHALLENGE**: ESTIMATE δ IN $\mathbb{Z}/2^n\mathbb{Z}$, $PSL(2,p)$, ...

(2) FREE GROUPS (STANDARD GENS)

③ $\pi_1(S_2) = \langle a,b,c,d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} \rangle$