2025-02-06 SAULSCHLEIMER MA4H4 LECTURE 15

FIX MISTAKE IN DEF of END EQUIV:

SUPPOSE $\alpha, \beta \subset \Gamma_S$ ARE GEOD. RAYS. SAY α, β ARE END EQUIVALENT IF, FOR ALL $n \geq 0$, WE HAVE THE INFINITE COMP'TS of $\alpha - B_S(n)$ AND $\beta - B_S(n)$ CONTAINED IN THE SAME COMP'T of $\Gamma_S - B_S(n)$.

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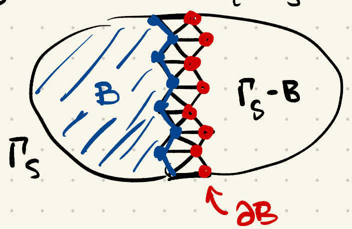
(1) BOUNDARIES: SUPPOSE $B \subset \Gamma_S$. WE DEFINE THE (VERTEX) BOUNDARY of B TO BE $\partial B = \{ h \in G \mid d_S(B, h) = 1 \}$.

HERE $d_S(B,h) = \text{MIN} \{ d_S(b,h) \mid b \in B \}$.

PICTURE:



(2) RECAP: (G,S) TWO ENDED. GIVEN $C_0 = C_S(n)$ AND $G = A^* \sqcup C_0 \sqcup A$. NOTE A^*, C_0, A ALL CONNECTED.

EXERCISE: $\partial A, \partial A^* \subset C_0$.

EXERCISE: [SEPARATORS] SUPPOSE $B \subset \Gamma_S$ CONNECTED. SUPPOSE $B \cap C_0 = \emptyset$. THEN $B \subset A$ (SO $B \cap A^* = \emptyset$)

OR $B \subset A^*$ (SO $B \cap A = \emptyset$).

PICK $g \in A \cap H$ WITH $|g|_S \geq 2n+1$.

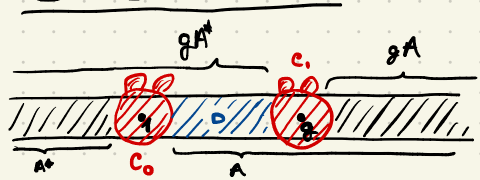
DEFINE $C_1 = g \cdot C_0$.

EXERCISE: $C_0 \cap C_1 = \emptyset$.

STEP (3): SHOW $A^* \cup C_0 \subset gA^*$

AND $gA \cup C_1 \subset A$

DESIRED PICTURE:



NOW: C_0 SEPARATES A FROM A^* AND $C_0 \cap C_1 = \emptyset$. ALSO, C_1 IS CONNECTED. THUS $C_1 = g \cdot C_0$ LIES IN EITHER A OR A^*. SINCE $g \in C_1$ WE HAVE $C_1 \subset A$. SO $C_1 \cap A^* = \emptyset$.

LIKEWISE: C_1 SEPARATES gA FROM gA^* AND A^* IS CONNECTED. SO A^* LIES IN EITHER gA^* OR gA.

FIX A GEOD. RAY $\alpha^* \subset A^*$. SO $g\alpha^*$ IS END-EQUIV TO α^* [BECAUSE $g \in H = \mathrm{KER}(\mathrm{ACTION})$]. PICK $m \in \mathbb{N}$ SO THAT $C_0 \cup C_1 \subset B_S(m)$. SO INF COMP'TS OF $\alpha^* - B_S(m)$ AND $g \cdot \alpha^* - B_S(m)$ LIE IN SAME COMP'T OF $\Gamma_S - B_S(m)$.
SO SOME POINT OF A^* IS CONNECTED TO SOME POINT OF gA^* IN THE COMPLEMENT OF C_1. WE DEDUCE $A^* \subset g \cdot A^*$.

<u>CLAIM</u>: $A^* \cup C_0 \subset gA^*$. IN PARTICULAR, $A^* \neq g \cdot A^*$.
SINCE $\partial A, \partial A^* \subset C_0$, FIND $\partial(gA), \partial(gA^*) \subset gC_0 = C_1$.
<u>EXERCISE</u>: $A^* \cup \partial A^* \subset gA^* \cup \partial(gA^*)$.
NOW $C_0 \cap C_1 = \emptyset$. THUS $\partial A^* \cap \partial(gA^*) = \emptyset$. SO $\partial A^* \subset gA^*$.
THUS $C_0 \cap gA^* \neq \emptyset$. BUT C_1 SEPARATES gA^* FROM gA, $C_0 \cap C_1 = \emptyset$, AND C_0 CONNECTED. SO DEDUCE $C_0 \subset gA^*$.
THUS $\boxed{A^* \cup C_0 \subset gA^*}$.

<u>COROLLARY</u>: $\boxed{gA \cup C_1 \subset A}$ <u>PROOF</u>: TAKE COMPLEMENTS. $\square$

<u>CLAIM</u>: $g^0 \in C_0$, ALSO $g^{-k} \in A^*$, $g^k \in A$ FOR ALL $k > 0$.
<u>PROOF</u>: $g \in A$ BY CONSTRUCTION. SINCE $gA \subset A$, $g^k \in A$ FOR ALL $k > 0$. SINCE $C_0 \subset gA^*$, WE HAVE $g^{-1} \in A^*$.
SINCE $A^* \subset gA^*$ WE HAVE $g^{-1}A^* \subset A^*$, $g^{-k} \in A^*$ FOR ALL $k > 0$

$\square$

COROLLARY: g HAS INF ORDER. $\square$

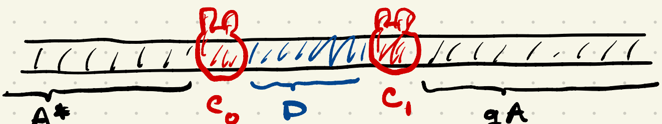
DEFINE $F = gA^* - A^*$.

EXERCISE: $F \subset C_S(n + |g|_S)$ SO F IS FINITE.

CLAIM: $G = \bigcup_{k\in\mathbb{Z}} g^k F$

PROOF: LET $D = gA^* \cap A = F - C_0$. SO $\Gamma = A^* \sqcup C_0 \sqcup D \sqcup C_1 \sqcup gA$ IN ORDER.

PICTURE



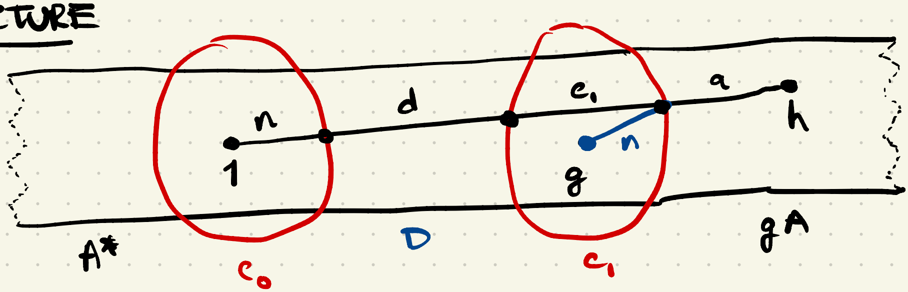
SINCE $C_0 \cap C_1 = \emptyset$ WE HAVE $d_S(a,b) \geq 1$ FOR ALL $a \in C_0$, $b \in C_1$.

THUS, ANY GEODESIC FROM 1_G TO $h \in gA$ HAS A SEGMENT OF LENGTH

- n IN C_0
- $d \geq 1$ IN D
- $c_1 \geq 0$ IN C_1 [AT LEAST A VERTEX]
- $a = |h|_S - n - d - c_1$ IN gA

SO
$$d_S(g,h) \leq n + |h|_S - n - d - c_1 = |h|_S - d - c_1 \leq |h|_S - 1.$$

PICTURE



THAT IS: $|g^{-1}h|_S \leq |h|_S - 1$, FOR ANY $h \in gA$.

SUBCLAIM: $A \subset \bigcup_{k \geq 0} g^k F$.

PROOF: $D \subset F$, $C_1 \subset g \cdot F$. ALSO $A = D \cup C_1 \cup gA$. SO SUFFICES TO PROVE $gA \subset \bigcup_{k \geq 0} g^k F$. INDUCT ON $|h|_S$ $\square$.

THUS: $[G, \langle g \rangle] = \mathrm{CARD}(F)$. $\square$ THEOREM.

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(3) MORE

THEOREM: [HOPF-FREUDENHAL] SUPPOSE G FIN. GEN. THEN $e(G) = 0, 1, 2$, OR ∞.

CHALLENGE: PROVE THIS.

THEOREM [STALLINGS]: SUPPOSE $e(G) \geq 2$. THEN G SPLITS (AS FREE PROD OR HNN EXT) OVER A FINITE GROUP.

EX: $\mathbb{Z}$, D_∞, $F(a,b)$, $\mathbb{Z}^2 * \mathbb{Z}$ ...

SEE ALSO: BASS-SERRE THY [TREES BY SERRE]

TOPOLOGY of GRAPHS [STALLINGS]

JSJ DECOMPOSITIONS of GROUPS [SCOTT, OTHERS]