2025-02-05 SAUL SCHLEIMER MA4H4 LECTURE 14

① SET of ENDS.

(*) WAS WRONG IN LECTURE – FIXED HERE.

RECALL $\mathcal{R}_S = \{$ GEOD RAYS IN $\Gamma_S \}$. ALSO $\alpha, \beta \in \mathcal{R}_S$ ARE (*) **END EQUIV** IF, FOR ALL $n \geq 0$, THE INF COMPONENTS of $\alpha - B_S(n)$ AND $\beta - B_S(n)$ LIE IN THE SAME COMPONENT of $\Gamma_S - B_S(n)$. WRITE $\alpha \overset{E}{\sim} \beta$.

DEFINE: $\text{ENDS}(G,S) = \mathcal{R}_S / \text{END-EQUIV}$.

EXERCISE: SUPPOSE $A \in \mathcal{E}_S(n)$. THEN THERE IS A GEODESIC RAY IN Γ_S CONTAINED IN A. [DOES NOT FOLLOW IMMEDIATELY FROM PREVIOUS EXERCISE!]

PROP: $e(G,S) = \text{CARD}(\text{ENDS}(G,S))$ [WHERE WE USE "∞" FOR ANY INF. CARDINAL].

PROOF: SUPPOSE $A, A' \in \mathcal{E}_S(n)$ DISTINCT. THEN THE GEOD RAYS $\alpha \subset A$, $\alpha' \subset A'$ [EXERCISE] ARE NOT END EQUIV. THUS $e(G,S) \leq \text{CARD}(\text{ENDS})$. ON THE OTHER HAND, SUPPOSE $\alpha_1, \alpha_2, \ldots \alpha_N$ ARE N INEQUIV GEOD RAYS. THEN THERE IS SOME M SO THAT NO TWO of $\alpha_i - B_S(m)$, $\alpha_j - B_S(m)$ LIE IN SAME COMPONENT of $\Gamma_S - B_S(m)$. SO $e(G,S) \geq \text{CARD}(\text{ENDS})$. □

② ACTING ON ENDS.

NOTE G ACTS ON $\mathcal{R}_S$ AND PRESERVES END-EQUIVALENCE. SO G ACTS ON ENDS(G,S) AS A SET — THAT IS VIA BIJECTIONS. LET $G \longrightarrow \text{AUT}(\text{ENDS}(G,S))$ BE THE INDUCED HOMOMORPHISM.

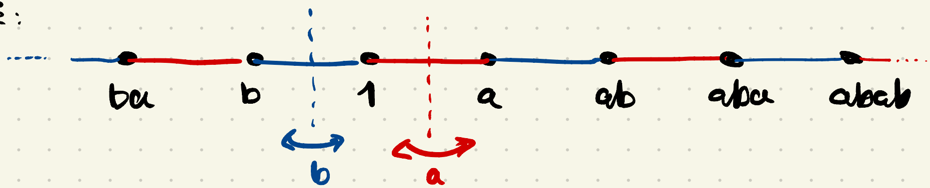
[IN FACT ENDS(G,S) HAS A TOPOLOGY WHERE THE ACTION IS BY HOMEOMORPHISMS [HOPF-FREUDENTHAL]]

OF COURSE, IF $e(G) \leq 1$ THEN THE ACTION IS TRIVIAL.

EXAMPLE: $\mathbb{Z}$ ACTS TRIVIALLY ON ITS ENDS.

EXAMPLE: TAKE $G = D_\infty \cong \langle a, b \mid a^2 = b^2 = 1 \rangle$. Ex: [ACTION ON ENDS]

PICTURE:



THE ELEMENTS a, b ACT ON $\Gamma_S \cong \mathbb{R}$ VIA REFLECTIONS.

EXERCISE: EVERY $w \in F(a,b)$, WITH $w \neq e_G$ HAS TWO FIXED POINTS IN ENDS$(F(a,b))$ [WHICH IS HOMEOMORPHIC TO A CANTOR SET]

(3) TWO-ENDED GROUPS

THEOREM: [HOPF-FREUDENTHAL] $e(G) = 2$ IF AND ONLY IF G IS VIRTUALLY $\mathbb{Z}$.

PROOF: BACKWARDS DIRECTION FOLLOWS FROM EXERCISES.

FORWARD DIRECTION: SUPPOSE $e(G,S) = 2$.

STEP (1): SO AUT(ENDS(G)) $\cong \mathbb{Z}/2\mathbb{Z}$. DEFINE H BY:

$H = \mathrm{KER}(G \to \mathrm{AUT}(\mathrm{ENDS}))$. SO H HAS INDEX ONE OR TWO IN G.

EXERCISE: H MEETS EVERY 1-BALL $B_S(g, 1)$ IN Γ_S.

[IN PARTICULAR H IS INFINITE].

STEP (2): FIX $n \geq 0$ SO THAT $\mathcal{E}_S(n)$ HAS TWO ELEMENTS

SAY A AND A^*. SET $C_0 = C_S(n)$. SO $\Gamma_S = A^* \sqcup C_0 \sqcup A$.

PICTURE:

EXERCISE: $A \cap H$ IS INFINITE.

SO PICK ANY $g \in A \cap H$ WITH $|g|_S \geq 2n+1$.

EXERCISE: $C_0 \cap g \cdot C_0 = \emptyset$. SET $C_1 = g \cdot C_0$. So $C_0 \cap C_1 = \emptyset$.

DESIRED PICTURE: