2025-02-04 LECTURE 13 SAUL SCHLEIMER MA4H4

① TRYING AGAIN, WITH BETTER NOTATION.

G A GROUP, S, T FIN. GEN SETS. DEFINE $\Gamma_S = \Gamma(G,S)$.

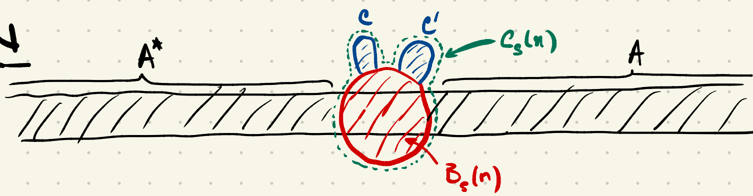
$B_S(n) = \{g \in G \mid |g|_S \leq n\}$ THE n-BALL; NOTE IT CONTAINS THE "SPHERE" of RADIUS n.

DEFINE $\mathcal{E}_S(n)$ TO BE THE SET of INF. CONN. COMPONENTS of $\Gamma_S - B_S(n)$. DEFINE Γ_T, $B_T(n)$, $\mathcal{E}_T(n)$ SIMILARLY.

IT WILL BE USEFUL TO "STORE" THE FINITE CONN. COMP'TS of $\Gamma_S - B_S(n)$ ALONG WITH THE BALL. THAT IS

DEFINE: $\mathcal{C}_S(n) = \Gamma_S - \bigcup_{A \in \mathcal{E}_S(n)} A = B_n(S) \cup \left\{ \begin{matrix} \text{FINITE CONN} \\ \text{COMP'TS of} \\ \Gamma_S - B_S(n) \end{matrix} \right\}$

CARTOON



FOR ANY FINITE $C \subset \Gamma_S$ WE DEFINE ITS DIAMETER

$\mathrm{DIAM}(C) = \mathrm{MAX}\{d_S(g,h) \mid g, h \in C\}$.

② ENDS NEVER DIE (SECOND ATTEMPT)

SET $C = \mathrm{MAX}\{|s|_T : s \in S\}$ RECALL $|g|_T \leq C \cdot |g|_S$

$D = \mathrm{MAX}\{|t|_S : t \in T\}$ $\quad |g|_S \leq D \cdot |g|_T$

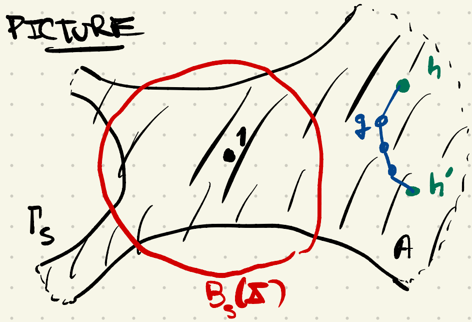
RECALL AS WELL THAT Γ_S, Γ_T HAVE THE SAME VERTICES.

LEMMA: FIX $X \in \mathbb{N}$ AND $Y \geq C \cdot X + C \cdot D$. THEN INCLUSION INDUCES A SURJECTION $\iota : \mathcal{E}_T(Y) \twoheadrightarrow \mathcal{E}_S(X)$.

PROOF: FIX $A \in \mathcal{E}_S(X)$. NOTE $A - B_T(Y)$ IS INFINITE AND $\mathcal{E}_T(Y)$ IS FINITE. SO $A \cap B$ IS INFINITE FOR SOME $B \in \mathcal{E}_T(Y)$. WE NOW MUST SHOW THAT THE VERTICES of B LIE IN A. SO, FIX ANY

$h \in A \cap B$. SUPPOSE (h,h') IS ANY ADJACENT EDGE OF B. SO $d_T(h,h') = 1$. THUS $d_S(h,h') \le D$. FIX AN EDGE PATH α FROM h TO h' IN Γ_S WITH $|\alpha| \le D$.

PICTURE



CLAIM: $\alpha \subset A$.

PROOF: SINCE A IS CONNECTED, AND $h \in A$, IT SUFFICES TO SHOW, FOR EACH VERTEX $g \in \alpha$, THAT $g \notin B_S(X)$. SO, SUPPOSE FOR A CONTRADICTION THAT SOME $g \in \alpha$ LIES IN $B_S(X)$. SO $d_S(1,g) \le X$. NOTE $d_S(g,h) \le D$ BECAUSE $|\alpha| \le D$. THUS $d_S(1,h) \le X + D$. SO $d_T(1,h) \le C \cdot X + C \cdot D$. THUS $h \in B_T(Y)$ AND SO $h \notin B$. THIS IS THE CONTRADICTION PROVING THE CLAIM. $\square$

THUS $h' \in A$. SINCE B IS CONNECTED, INDUCTION PROVES THAT THE VERTICES OF B ARE CONTAINED IN A. SO $\iota : \mathcal{E}_T(Y) \longrightarrow \mathcal{E}_S(X)$ IS WELL-DEFINED AND SURJECTIVE. $\square$

COROLLARY: $e(G,S) = \lim_{n \to \infty} \text{CARD}(\mathcal{E}_S(n))$ IS WELL DEF.

PROOF: TAKE $S = T$, NOTE $C = D = 1$, AND THAT $C \cdot X + C \cdot D = X + 1$. SO $\mathcal{E}_S(X+1) \twoheadrightarrow \mathcal{E}_S(X)$ SURJECTS BY LEMMA. SO $\text{CARD}(\mathcal{E}_S(n))$ IS NON-DECREASING. $\square$

COROLLARY: $e(G,S) = e(G,T)$.

PROOF: $e(G,T) \ge e(G,S)$ AND $e(G,S) \ge e(G,T)$ BY LEMMA $\square$

EXERCISE: SUPPOSE $H < G$ IS FINITE INDEX. THEN $e(H) = e(G)$.

③ SET of ENDS. WITH G, S AS USUAL.

(*) CORRECTED DEFINITION.

DEFINE $\mathcal{R}_S = \{$ GEODESIC RAYS IN $\Gamma_S \}$.

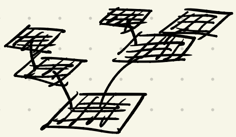
WE SAY $\alpha, \beta \in \mathcal{R}_S$ ARE END-EQUIVALENT IF, FOR ALL n, WE HAVE THAT THE INFINITE COMPONENTS (*) of $\alpha - B_S(n)$ AND $\beta - B_S(n)$ LIE IN THE SAME CONN COMPONENT of $\Gamma_S - B_S(n)$. WE USE $\alpha \overset{E}{\sim} \beta$ TO DENOTE THIS RELATION.

EXERCISE: PROVE $\overset{E}{\sim}$ IS AN EQUIV. RELATION.

DEFINE $\mathrm{ENDS}(G,S) = \mathcal{R}_S / \overset{E}{\sim}$, THE SET of ENDS of G

EXAMPLE: $\mathbb{Z}^2 * \mathbb{Z} \cong \langle a, b, c \mid aba^{-1}b^{-1} \rangle$

PICTURE:



HAVE ONE END FOR EACH COPY of $\mathbb{Z}^2$ (COUNTABLY MANY of THESE) AND ALSO HAVE ONE END FOR EACH GEOD RAY (BASED AT 1_G) THAT CROSSES INF MANY COPIES of $\mathbb{Z}^2$ (UNCOUNTABLY MANY of THESE).