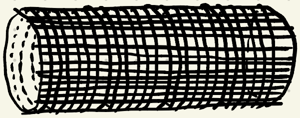
2025-01-30 LECTURE 12 MA4H4 SAUL SCHLEIMER

[SUPPORT CLASS IN MB2.22]

(1) ENDS, EVENTUALLY.

EXAMPLE: $G = \mathbb{Z} \times \mathbb{Z}/100\mathbb{Z}$ HAS TWO ENDS, BUT THIS IS NOT SEEN UNTIL RADIUS ≈ 50.

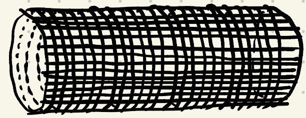
PICTURE:

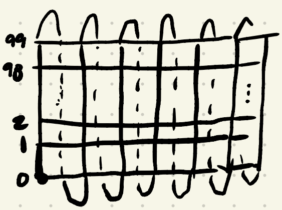


$B(n)$ FOR $n < 50$ IS ISOMORPHIC (AS A GRAPH) TO A BALL IN $\mathbb{Z}^2$.

EXAMPLE: TAKE $G \cong \mathbb{Z}$ AND $S = \{1, 100\}$. LET $\Gamma = \Gamma(G,S)$.

THEN Γ IS ALMOST A COPY OF THE ABOVE:





HAS HELIX INSTEAD OF PRODUCT BUT OTHERWISE "ALMOST ISOMETRIC".

[ENDS DO NOT BEHAVE SIMPLY UNDER GROMOV HAUSDORFF LIMITS]

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(2) ENDS NEVER DIE

[BETTER VERSION NEXT WEEK]

FIX (G,S) AND DEFINE

$$e(G,S,n) = \mathrm{CARD}\left\{ \begin{array}{c} \text{INF. CONN COMPONENTS} \\ \text{of } \Gamma(G,S) - B(n) \end{array} \right\}.$$

LEMMA: SUPPOSE $n > m$. THEN $e(G,S,n) \geq e(G,S,m)$.

PROOF: SET $\Gamma = \Gamma(G,S)$. SUPPOSE $A_1, A_2, \ldots A_E$ ARE THE INF CONN. COMPONENTS of $\Gamma - B(m)$. SUPPOSE $B_1, B_2, \ldots, B_F$ ARE THE INF. CONN COMPONENTS of $\Gamma - B(n)$. THUS $\sqcup B_j \subset \sqcup A_i$

PICTURE:

SINCE B_j CONNECTED IT LIES IN SOME UNIQUE A_i. IF A_i CONTAINS NO B_j THEN

$A_i \subset B(n) - B(m)$. BUT $\mathrm{CARD}(B(m)) < \infty$, CONTRADICTION.

DEDUCE $F \geq E$. $\square$

THUS $e(G,S)$ WELL-DEFINED.

③ **INDEPENDENCE of GEN SET.**

PROP: SUPPOSE G IS A GROUP. SUPPOSE S,T ARE FINITE GEN. SETS. THEN $e(G,S) = e(G,T)$.

PROOF: SET $C = \max\{|s|_T : s \in S\}$

$D = \max\{|t|_S : t \in T\}$.

$\boxed{\textbf{LEMMA } |g|_T \leq C \cdot |g|_S \quad |g|_S \leq D \cdot |g|_T}$

FIX $n \in \mathbb{N}$. DEFINE $\boxed{\begin{array}{l} \mathbb{I} = C \cdot n + C \cdot D \\ \mathbb{X} = D \cdot \mathbb{I} \end{array}}$

LET $A_1 \cdots A_E$ BE THE INF CONN COMPONENTS of $\Gamma_S - B_S(n)$. SO $e(G,S) \geq E$. LET $B_1, B_2, \cdots B_F$ BE THE INF CONN COMP. of $\Gamma_T - B_T(\mathbb{I})$. SO $e(G,T) \geq F$.

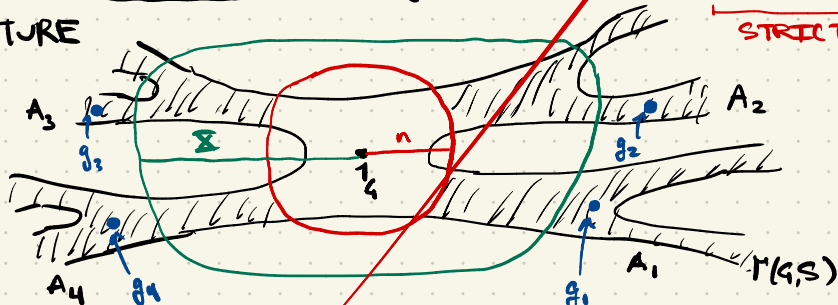
CLAIM: $F \geq E$.

NOTE THAT THIS IMPLIES $e(G,T) \geq E$, SO (TAKING n TO INF) IMPLIES $e(G,T) \geq e(G,S)$. A SYMMETRIC ARGUMENT GIVES $e(G,S) \geq e(G,T)$ AND SO GIVES THE PROPOSITION.

PROOF of CLAIM: FIX $g_i \in A_i$ WITH $|g_i|_S > \mathbb{X}$.

STRICT.

PICTURE



NOTE g_i EXIST BECAUSE A_i IS NOT CONTAINED IN $B_S(\mathbb{X})$.

SUBCLAIM: SUPPOSE $i \neq j$. THEN g_i, g_j LIE IN DISTINCT (INF, CONN) COMPONENTS of $\Gamma_T - B_T(Y)$.

NOTE THAT THIS PROVES THE CLAIM BECAUSE THERE ARE E DISTINCT g_i.

PROOF of SUBCLAIM: SET $g = g_i$, $h = g_j$.

SUBSUBCLAIM: $g, h \notin B_T(Y)$.

PROOF: TO OBTAIN A CONTRADICTION, SUPPOSE $g \in B_T(Y)$ THAT IS, $|g|_T \leq Y$. BY LEMMA, $|g|_S \leq D \cdot Y = X$. BUT $|g|_S > X$ BY HYPOTHESIS. $\square$

GOT HERE

BACK TO THE SUBCLAIM: FOR A CONTRADICTION SUPPOSE g, h LIE IN THE SAME (INF. CONN) COMPONENT B of $\Gamma_T - B_T(Y)$. SO FIX AN EDGE PATH $\beta \subset B$ FROM g TO h. LET (β_k) BE THE VERTICES of β.

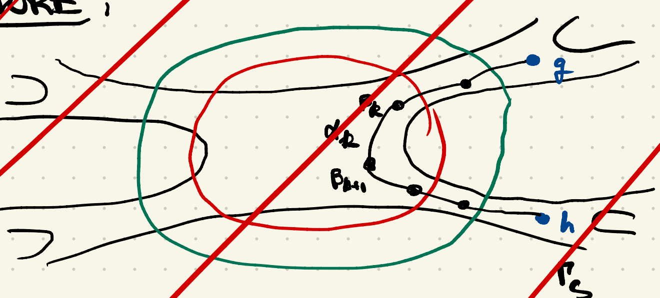
SO (i) $|\beta_k| > Y$ FOR ALL k. [STRICT]

WE USE β TO BUILD AN EDGE PATH $\alpha \subset \Gamma_S$. WE BEGIN WITH THE β_k. WE CONNECT β_k TO β_{k+1} VIA AN EDGE PATH α_k of LENGTH AT MOST D. SET $\alpha = \cup_k \alpha_k$.

SO: α CONNECTS g TO h, CONTAINS THE β_k AND β_k, β_{k+1} ARE CONN BY SUBARCS of LENGTH $\leq D$.

RECALL: g, h LIE IN DISTINCT COMPONENTS of $\Gamma_S - B_S(n)$

PICTURE:



SO α MEETS $B_S(n)$. THUS SOME β_k HAS $|\beta_k|_S \leq n + D$.

THUS $|\beta_R|_T \leq C\cdot n + C\cdot D$. BUT $|\beta_R|_T > I$ BY HYPOTHESIS. THIS PROVES THE SUBCLAIM, THUS THE CLAIM, AND THUS THE PROPOSITION. $\square$