2025-01-29 LECTURE 11 MA4H4 SAULSCHLEIMER

[SUPPORT CLASS MB2.22.]

(1) VIRTUALITY

EXERCISE: SUPPOSE $H < G$ FIN. INDEX

THEN: (1) G FIN. GEN IF AND ONLY IF H FIN. GEN.

(2) $\gamma_{G,S} \approx \gamma_{H,T}$ [COMPARABLE VOLUME GROWTHS]

DEF. SUPPOSE G IS A GROUP. SUPPOSE $H < G$ IS A FINITE INDEX SUBGROUP. SUPPOSE H HAS SOME PROPERTY P. THEN WE CALL G VIRTUALLY P.

VIRTUALLY P IMPLIES P	VIRTUALLY P DOES NOT IMPLY P
FINITE, INFINITE	TRIVIAL
OF POLYNOMIAL GROWTH	ABELIAN, SIMPLE
FIN. GENERATED	NILPOTENT, SOLVABLE
FIN. PRESENTED.	FREE
SOLVABLE WORD PROB.	TORSION-FREE

HERE ARE THE THEOREMS FROM YESTERDAY:

THEOREM [HIRSCH] SUPPOSE G FIN. GEN AND NILPOTENT. THEN G IS VIRTUALLY TORSION FREE.

THEOREM [JENNINGS] SUPPOSE G FIN GEN AND NILPOTENT. THEN G VIRTUALLY EMBEDS IN $H_d(\mathbb{Z})$ [FOR SOME d].

THEOREM [GROMOV] SUPPOSE G FIN GEN WITH GROWTH AT MOST POLYNOMIAL. THEN G VIRTUALLY NILPOTENT.

THERE IS INTEREST IN "ELEMENTARY" VERSIONS of GROMOV'S

THEOREM. AS AN EXAMPLE

THEOREM [WILKIE - VAN DEN DRIES 1984] SUPPOSE G FIN GEN. SUPPOSE THERE IS SOME n WITH $\gamma_{G,S}(n) < \frac{1}{2}(n+2)(n+1)$. THEN G IS VIRTUALLY CYCLIC.

OPEN: GIVE EXPLICIT GROWTH CONDITIONS ENSURING G IS VIRTUALLY $\mathbb{Z}^2$.

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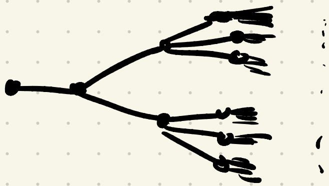
② GEODESIC RAYS AND LINES

DEF: A GRAPH Γ IS LOCALLY FINITE IF, FOR ALL $v \in V(\Gamma)$ v MEETS ONLY FINITELY MANY EDGES.

EX:



AT LEVEL k THE DEGREES ARE ALL $k+1$.

LEMMA: SUPPOSE Γ IS A CONNECTED, INFINITE, LOCALLY FINITE GRAPH. THEN Γ CONTAINS A GEODESIC RAY.

PROOF: EXERCISE. $\square$

LEMMA: SUPPOSE G IS A GROUP, S A FIN. GEN SET. SUPPOSE G IS INFINITE. THEN $\Gamma = \Gamma(G,S)$ CONTAINS A GEODESIC LINE.

PROOF: EXERCISE. $\square$

③ ENDS of GROUPS

WE GIVE TWO RELATED APPROACHES.

DEF: SUPPOSE G IS A GROUP. SUPPOSE $S \subset G$ IS A FIN GEN SET. SET $\Gamma = \Gamma(G,S)$. DEFINE

$$e(G,S) = \lim_{n \to \infty} \text{CARD} \left\{ \begin{array}{c} \text{INF. CONN. COMPONENTS} \\ \text{of } \Gamma - B(n) \end{array} \right\}$$

THIS IS THE NUMBER of ENDS of G (RELATIVE TO S).

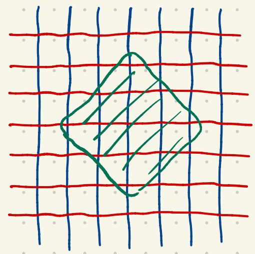
PROPOSITION: $e(G,S)$ IS INDEPENDENT of S.

PROOF: TOMORROW.

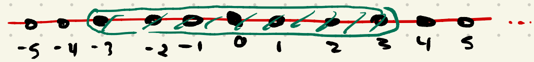
EXAMPLES

(1) $e(G,S) = 0$ IFF G IS FINITE.

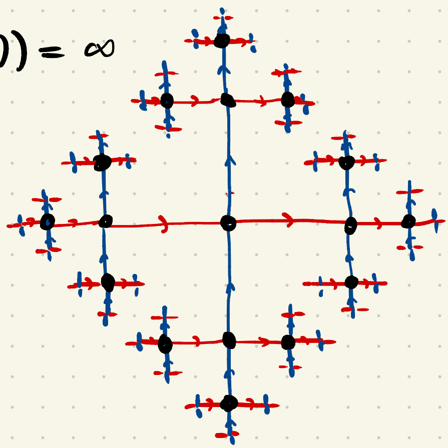
(2) $e(\mathbb{Z}^2, \text{STD}) = 1$. **PICTURE**:



(3) $e(\mathbb{Z}, \text{STD}) = 2$



(4) $e(F(a,b)) = \infty$



MORE ON ENDS:

THEOREM [HOPF 1943, FREUDENTHAL 1945]

① SUPPOSE G FIN GEN. THEN $e(G) = 0, 1, 2,$ OR ∞

② $e(G) = 2$ IF AND ONLY IF G IS VIRTUALLY $\mathbb{Z}$.

THEOREM [STALLINGS, 1971] IF $e(G) > 1$ THEN G **SPLITS** OVER SOME FINITE SUBGROUP $F < G$.