2025-01-28 LECTURE 10 SAUL SCHLEIMER MA444

SUPPORT CLASS IN MB2.22. THIS WEEK

(1) CHANGING GENERATORS

PROPOSITION: SUPPOSE G IS A GROUP. SUPPOSE S, T ARE FINITE GEN SETS FOR G. SET $\gamma_S = \gamma_{G,S}$, $\gamma_T = \gamma_{G,T}$. THEN THERE ARE CONSTANTS C, D SO THAT

$$\text{FOR ALL } n \in \mathbb{N} \quad \begin{cases} \gamma_S(n) \leq \gamma_T(C \cdot n) \\ \gamma_T(n) \leq \gamma_S(D \cdot n) \end{cases}$$

PROOF SET $C = \max\{|s|_T : s \in S\}$. BY LEMMA HAVE $d_T(1_G, g) \leq C \cdot d_S(1_G, g)$. THUS $B_S(n) \subset B_T(C \cdot n)$. SO $\gamma_S(n) \leq \gamma_T(C \cdot n)$, AS DESIRED. $\square$

LEMMA: FIX $d \in \mathbb{N}$. SET $G = \mathbb{Z}^d$ AND S A FINITE GEN SET. THEN, WITH $\gamma = \gamma_{G,S}$, THERE ARE CONSTANTS C, D SO THAT

$$C \cdot n^d \leq \gamma(n) \leq D \cdot n^d$$

PROOF: EXERCISE (LAST TIME) AND PROPOSITION ABOVE $\square$

COROLLARY: $\mathbb{Z}^p \cong \mathbb{Z}^q$ IF AND ONLY IF $p = q$.

PROOF: EXERCISE (LAST TIME) AND PROPOSITION ABOVE $\square$

EXERCISES:

(1) FOR $G = H_3(\mathbb{Z})$ $\gamma(n) \sim n^4$.

(2) FOR $G = BS(1,2)$ $\gamma(n) \sim \mathrm{EXP}(n)$

(3) IF $F(a,b) < G$ $\gamma(n) \sim \mathrm{EXP}(n)$.

SO FAR: WE HAVE FOUND GROUPS WITH POLY-GROWTH AND EXP-GROWTH.

EXERCISE: γ_G IS SUBLINEAR IFF G IS FINITE.

SO: $\gamma(n) \sim \sqrt{n}$ OR $\log(n)$ IS NOT POSSIBLE.

NOTE: BY EXERCISE $\gamma(m) \leq (2|S|+1)^m$ SO SUPEREXP GROWTH NOT POSSIBLE.

DEF: SAY G (FIN GEN) HAS γ_G SUPERPOLY BUT SUBEXP [EG $\exp(\sqrt{n})$]. THEN CALL G A GROUP of INTERMEDIATE GROWTH.

THEOREM [GRIGORCHUK, 1984] THERE IS A GROUP G of INTERMEDIATE GROWTH.

OPEN: CAN A FIN. PRES GROUP HAVE INTERMED. GROWTH?

(2) NILPOTENT GROUPS

DEF: SUPPOSE $A, B < G$ ARE SUBGROUPS. THEN $[A,B] < G$ IS THE SUBGROUP GEN BY $\{[a,b] \mid a \in A, b \in B\}$.

DEF: THE LOWER CENTRAL SERIES (G_i) FOR G IS

$$G_0 = G$$
$$G_{i+1} = [G_i, G]$$

DEF: WE SAY G IS NILPOTENT IF $G_k = 1$ FOR SOME k. THE FIRST SUCH k IS CALLED THE NILPOTENCY CLASS of G.

EXAMPLES

(1) 1 IS NILPOTENT of CLASS ZERO

(2) ABELIAN GROUPS ARE NILPOT. of CLASS ≤ 1

(3) $H_3(\mathbb{Z})$ IS NILPOT. of CLASS 2.

(4) $F(a,b)$ IS NOT NILPOTENT.

EXERCISE: DETERMINE IF $BS(1,2)$ IS NILPOTENT.

EXERCISE: DEFINE $H_d(\mathbb{Z}) < SL_d(\mathbb{Z})$ TO BE THE GROUP of UPPER TRIANG. MATRICES WITH ONES ON DIAGONAL. SHOW $H_d(\mathbb{Z})$ NILPOTENT of CLASS $d-1$. [FOR $d \geq 1$]

EXERCISE: $H_d(\mathbb{Z})$ IS TORSION FREE.

EXERCISE SUPPOSE G NILPOTENT of CLASS d. SUPPOSE $H<G$. THEN H NILPOTENT of CLASS $\leq d$.

THEOREM [HIRSCH, 1938] SUPPOSE G IS FIN GEN AND NILPOTENT. THEN THERE IS SOME $H<G$ WHICH IS TORSION FREE AND FINITE INDEX.

THEOREM [JENNINGS 1955] SUPPOSE G FIN GEN AND TORSION-FREE. THEN G EMBEDS IN $H_d(\mathbb{Z})$ FOR SOME $d \in \mathbb{N}$.

THEOREM [BASS 1972, GUIVARC'H 1973]. SUPPOSE G FIN GEN AND NILPOTENT. THEN THERE ARE C, D, d SO THAT

$$C \cdot n^d \leq \gamma_G(n) \leq D \cdot n^d \quad \text{FOR ALL } n \in \mathbb{N}.$$

[IN FACT $d = \sum_1 k \cdot \mathrm{RANK}(G_{k-1}/G_k)$]

③ THE "CLASSIFICATION" of GROUPS of POLYNOMIAL GROWTH

THEOREM [GROMOV, 1981] SUPPOSE G FIN GEN. SUPPOSE THAT THE VOLUME GROWTH of G IS AT MOST POLYNOMIAL.
THEN: THERE IS $H<G$ of FIN INDEX SO THAT H IS NILPOTENT.

THUS: GROWTH "BETWEEN" POLYNOMIALS (LIKE $n\log(n)$ OR $n^{7/3}$ ...) IS NOT POSSIBLE FOR FIN. GEN GROUPS.