2025-01-23 SAUL SCHLEIMER MA4H4 LECTURE 9

## ① SHORT LEX REPS.

FIX $(G,S)$ WITH $|S|<\infty$. FIX ORDER ON $S\cup S^{-1}$ AND EXTEND TO GET $<_{sl}$ SHORTLEX ORDER ON $(S\cup S^{-1})^*$. SUPPOSE $g\in G$. WE DEFINE

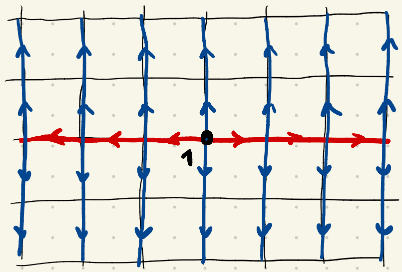
$$\min \{ u \in (S\cup S^{-1})^* \mid u =_G g \}$$

TO BE THE SHORTLEX REPRESENTATIVE of $g$.
THAT IS, FIRST PAY ATTENTION TO LENGTH AND THEN USE LEX ORDERING TO "BREAK TIES".

EXERCISE. ANY SUBWORD of A SHORTLEX REP IS ITSELF A SHORTLEX REPRESENTATIVE.

COROLLARY: THE EDGE PATHS IN $\Gamma(G,S)$ WITH LABEL A SHORTLEX REP FORM A SPANNING TREE. $\square$

EXAMPLE: FOR $\mathbb{Z}^2$ WITH $a<a^{-1}<b<b^{-1}$



EXERCISE: DRAW THE SPANNING TREES FOR THE OTHER ORDERS (THERE ARE NOT 24 of THEM!)

EXERCISE: DRAW THE SPANNING TREE FOR $K=\langle a,b \mid aba^{-1}=b^{-1}\rangle = BS(1,-1)$ WITH $a<a^{-1}<b<b^{-1}$.

## ② VOLUME GROWTH: FIX $(G,S)$ WITH $|S|<\infty$. SET $\Gamma=\Gamma(G,S)$.

DEFINE: $B_S(g,R)=\{h\in G \mid d_S(g,h)\le R\}$, THE BALL of RADIUS $R$ ABOUT $g$ IN $\Gamma(G,S)$.

NOTE: $g\cdot B_S(1_G,R)=B_S(g,R)$. SO:

**DEFINE**: $\gamma = \gamma_{G,S} : \mathbb{N} \longrightarrow \mathbb{N}$ BY $\gamma(R) = \mathrm{CARD}(B_S(1,R))$.

THIS IS THE VOLUME GROWTH of $G$ (WITH RESPECT TO $S$).

**EXERCISES**: SET $\gamma = \gamma_{G,S}$.

① FOR $m,n \in \mathbb{N}$ WE HAVE $\gamma(m+n) \leq \gamma(m) \cdot \gamma(n)$.

[HINT: USE SHORTLEX REPS.]

② COROLLARY: $\gamma(m) \leq (2|S|+1)^m$.

[SO EXP-GROWTH IS MAXIMUM POSSIBLE]

③ $G$ FINITE IFF $\gamma$ BOUNDED.

④ $G$ FINITE IFF $\gamma(n) = \gamma(n+1)$ FOR SOME $n \in \mathbb{N}$.

**EXAMPLES**:

① FOR $G = \mathbb{Z}$, $S = \{1\}$, $\gamma(n) = 1 + 2n$.

② FOR $G = \mathbb{Z}^2$ $S = \{(1,0),(0,1)\}$, $\gamma(n) = 1 + 2n + 2n^2$.

DEFINE $D(d,n) = \gamma_{\mathbb{Z}^d}(n)$ [USUAL GEN SET]

| $d \backslash n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 3 | 5 | 7 | 9 | 11 |
| 2 | 1 | 5 | 13 | 25 | 41 | 61 |
| 3 | 1 | 7 | 25 | 63 | $\square$ | $\square$ |
| 4 | 1 | 9 | $\square$ | $\square$ | $\square$ | $\square$ |

**EXERCISE**: FILL IN THE MISSING SQUARES.
[SEE OEIS: A008288]

**OPEN**: GIVE A "CLOSED FORM" FOR $D(d,n)$.

PROVE: ① $D(d,n) = D(n,d)$
② $D(d,n) = D(d-1,n) + D(d-1,n-1) + D(d,n-1)$

**EXERCISE**: FIX $d$. PROVE THERE IS A CONSTANT $C = C(d)$ SO THAT

$$\frac{1}{C} \cdot n^d \leq D(d,n) \leq C \cdot n^d.$$

③ **$F(S)$ IS THE BIGGEST**: FIX $S$ AND DEFINE $\gamma_G = \gamma_{G,S}$, $\gamma_{F(S)} = \gamma_{F(S),S}$

**LEMMA**: FOR ALL $n$ $\gamma_G(n) \leq \gamma_{F(S)}(n)$. ALSO $\gamma_G = \gamma_{F(S)}$ IFF $G \cong F(S)$

**PROOF**: RECALL $\bar{\varphi} : F(S) \to G$ INDUCES A COVERING of $\Gamma(G,S)$

BY $\Gamma(F(S),S)$. SO $\bar{\varphi}$ SENDS $B_{F(S),S}(\varepsilon_S,n)$ ONTO $B_{G,S}(1_G,n)$.

SO $\gamma_G \leq \gamma_{F(S)}$. NOW, IF $\gamma_G = \gamma_{F(S)}$ THEN $\Gamma(G,S)$ IS A TREE.

SO $G$ HAS NO RELATORS. $\square$

WE CAN SLIGHTLY IMPROVE THE ESTIMATE ON $\gamma_{F(S)}$.

<u>LEMMA</u>: $\gamma_{F(S)}(n) = 1 + 2|S|\left((2|S|-1)^n - 1/2|S|-2\right) \sim (2|S|-1)^n$

<u>PROOF</u>: TAKE $k = 2|S|-1$. SO

$$\gamma(n) = \overset{0}{1} + 2|\overset{1}{S}| + 2|\overset{2}{S}|\cdot k + 2|S|\overset{3}{\cdot} k^2 + \cdots + 2|S|\overset{n}{\cdot} k^{n-1}$$

$$= 1 + 2|S|\left(1 + k + k^2 + \cdots + k^{n-1}\right)$$

$$= 1 + 2|S|\cdot(k^n - 1)/(k-1) \qquad \square$$