2025-01-22 LECTURE 8 MA4H4 SAUL SCHLEIMER

ROOMS: 3-6 MB2.22
7-8 MB2.24
9 MB0.08
10 C1.14

① WHAT CAN BE COMPUTED?

SOME POSSIBILITIES:

A) GIVEN $G = \langle S \mid R \rangle$ WE CAN COMPUTE $G^{AB} = G/[G,G]$.

B) GIVEN F (FINITE GROUP) AND $G = \langle S \mid R \rangle$ CAN DECIDE IF G SURJECTS F.

c) GIVEN $d \in \mathbb{N}$ AND $G = \langle S \mid R \rangle$ CAN DECIDE IF THERE IS A NONTRIVIAL HOMOMORPHISM $G \longrightarrow SL_d(\mathbb{C})$

d) GIVEN $G = \langle S \mid R \rangle$ CAN DECIDE IF G SURJECTS THE FREE GROUP OF RANK TWO. [MAKANIN, 1982]

② CHANGING GENERATORS

LEMMA: SUPPOSE $S, T \subset G$ ARE FINITE GEN SETS FOR G. THEN THERE ARE CONSTANTS C, D SO THAT, FOR ALL $g, h \in G$:

$$d_T(g,h) \leq C \cdot d_S(g,h)$$

$$d_S(g,h) \leq D \cdot d_T(g,h).$$

PROOF: SET $C = \max\{ |s|_T : s \in S \}$.

FIX $g, h \in G$. FIX EDGE PATH $\gamma : [0,n] \longrightarrow \Gamma(G,S)$ WITH $\gamma(0) = g$, $\gamma(n) = h$, $n = d_S(g,h)$. SUPPOSE $u \in F(S)$ IS THE LABEL OF γ. LET $v_i \in F(T)$ BE A WORD SO THAT

① $|v_i| \leq C$

② $v_i =_G u_i$

DEFINE δ TO BE THE EDGE PATH IN $\Gamma(G,T)$ WITH $\delta(0) = g$ AND LABEL $v_0 v_1 \cdots v_{n-1}$. SO δ ENDS AT h.

ALSO $|\delta| \le c \cdot n = c \cdot d_S(g,h)$. SO $d_T(g,h) \le c \cdot d_S(g,h)$ $\blacksquare$

WE SAY THE METRICS d_S AND d_T ARE <u>COMPARABLE</u>

<u>DEF</u>: SUPPOSE $H < G$ ARE GENERATED BY T AND S, BOTH FINITE. WE SAY H IS <u>UNDISTORTED</u> IN G IF THERE IS A CONSTANT c SO THAT, FOR ALL $g, h \in H$

$$d_T(g,h) \le c \cdot d_S(g,h) \qquad [\text{"NO SHORTCUTS"}]$$

[OPPOSITE INEQUALITY AS IN LEMMA]

<u>EXERCISES</u>:
1. $\mathbb{Z} < \mathbb{Z}^m$ UNDISTORTED.
2. $\mathbb{Z} < F(S)$ UNDISTORTED.
3. $\langle b \rangle < \langle a, b \mid aba^{-1} = b^2 \rangle$ DISTORTED.
4. $\left\langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle < H_3(\mathbb{Z})$ DISTORTED.

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(3) <u>GEODESICS</u>: SUPPOSE (X, d_X) IS A METRIC SPACE. SUPPOSE $I \subset \mathbb{R}$ IS CONNECTED AND CLOSED [AN INTERVAL, RAY, OR LINE]. WE CALL $\gamma : I \to X$ A <u>GEODESIC</u> IF

$$d_X(\gamma(a), \gamma(b)) = |b - a| \qquad \text{FOR ALL } a, b \in I.$$

[DEDUCE γ IS CONTINUOUS]. NOTE: IF $J \subset I$ IS CONNECTED, CLOSED THEN $\gamma|J$ IS AGAIN A GEODESIC.

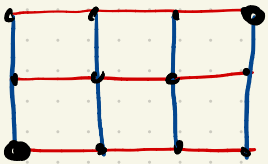
<u>DEF</u>: SUPPOSE $\gamma : [0, n] \to \Gamma(G, S)$ IS AN EDGE PATH AND IS GEODESIC. CALL THE LABEL of γ A <u>GEODESIC WORD</u>

<u>THUS</u>: SUBWORDS of GEODESIC WORDS ARE GEODESIC WORDS.

<u>ALSO</u>: GEOD. WORDS ARE REDUCED.

<u>SADLY</u>: GEODESIC WORDS NEED NOT BE UNIQUE.

EXAMPLE IN $\mathbb{Z}^2$:



$$\left.\begin{matrix} a^3b^2,\ a^2bab,\ a^2b^2a \\ aba^2b,\ ababa,\ ab^2a^2 \\ ba^3b,\ ba^2ba,\ baba^2,\ b^2a^3 \end{matrix}\right\} 10$$

THIS CAUSES PROBLEMS SO:

4 SHORT LEX

DEF: FIX AN ORDER $<$ ON $S \cup S^{-1}$. WE EXTEND TO DEFINE FOR $u, v \in (S \cup S^{-1})^*$:

$u <_{sl} v$ IF

1. $|u| < |v|$ OR
2. $|u| = |v|$ AND $u = w * u'$, $v = w * v'$ WITH $u'_0 < v'_0$.

EXAMPLE: IF $a < b < c$ THEN

$$cb < a^3 < a^2b < ca^2 < a^4.$$

WE CALL $<_{sl}$ THE SHORTLEX ORDER ON $(S \cup S^{-1})^*$.