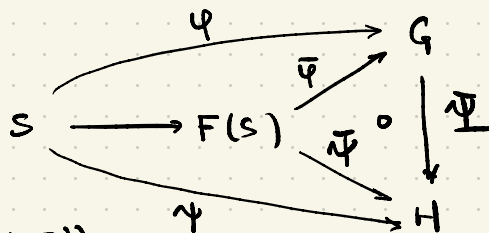


① UNIV. PROP. OF GROUP PRESENTATIONS: SUPPOSE  $G = \langle S | R \rangle$  IS A PRESENTATION, AND  $\bar{\varphi}: F(S) \rightarrow G$  IS THE INDUCED HOMOMORPHISM. SUPPOSE  $H$  IS A GROUP AND  $\psi: S \rightarrow H$  IS ANY FUNCTION. SUPPOSE  $R \subseteq \ker(\bar{\psi}: F(S) \rightarrow H)$ . THEN THERE EXISTS A UNIQUE HOMOMORPHISM  $\bar{\psi}: G \rightarrow H$  SO THAT  $\bar{\psi} = \bar{\psi} \circ \bar{\varphi}$ .

DIAGRAM:



THAT IS: " $G$  IS THE BIGGEST GROUP SATISFING THE RELATORS  $R$ ".

EXERCISE: PROVE THIS.

||

② WORD PROBLEM: [DEHN 1911]. NOW TAKE  $S$  FINITE. ALL OF THE CONSTRUCTIONS OF CAYLEY GRAPHS RELY ON SOLVING THE:

WORD PROBLEM FOR  $(G, S)$ : GIVEN  $u, v \in (S \cup S^{-1})^*$

DECIDE IF  $u =_G v$ . [IF  $\bar{\varphi} \circ r(u) = \bar{\varphi} \circ r(v)$ ]

THAT IS, FOR  $(G, S)$  FIXED FIND AN ALGORITHM THAT TAKES AS INPUT  $u$  AND  $v$  AND OUTPUTS

"YES" IFF  $u =_G v$

"NO" IFF  $u \neq_G v$ .

EQUIVANTLY INPUT IS  $w \in (S \cup S^{-1})^*$  AND DECIDE IF  $w =_G 1_G$ .

EXAMPLES: (1)  $(F(S), S)$ : REDUCE  $u, v$  TO OBTAIN

$r(u), r(v)$  AND CHECK EQUALITY AS STRINGS.

(2)  $(\mathbb{Z}^2, \{(1,0), (0,1)\})$ : ADD.

EXERCISE: GIVEN ALGORITHMS FOR  $WP(G, S)$ ,  $WP(H, T)$   
FIND AN ALGORITHM FOR  $WP(G \times H, S \cup T)$ .

CHALLENGE: SOLVE WORD PROBLEM FOR  $BS(1, 2)$ .

EXERCISE: SUPPOSE  $S, S'$  ARE GEN SETS FOR  $G$ . THEN  
 $WP(G, S)$  SOLVABLE IFF  $WP(G, S')$  SOLVABLE.

### ③ UNDECIDABILITY

THM [NOVIKOV, 1955] THERE IS A FIN. PRES GROUP  $G = \langle S | R \rangle$   
SO THAT  $WP(G, S)$  IS UNDECIDABLE.

COR: FOR THIS GROUP WE CANNOT BUILD  $\Gamma(G, S)$ .

THEOREM [ADIAN 1957, RABIN 1968] THERE IS NO ALGORITHM  
THAT, GIVEN FIN. PRES  $G = \langle S | R \rangle$ , DECIDES IF

- |           |                |                      |
|-----------|----------------|----------------------|
| $G$ IS    | ① TRIVIAL      | ⑤ RES. FINITE        |
| <u>OR</u> | ② FINITE       | ⑥ SOLVABLE WORD PROB |
|           | ③ COMMUTATIVE  | ⑦ SIMPLE             |
|           | ④ NILPOTENT    | ⑧ AUTOMATIC...       |
|           | ⑨ SOLVABLE     |                      |
|           | ⑩ TORSION-FREE | [SEE MILLER 1992]    |

NOTE: THIS DOES NOT MEAN GROUP THY IS "OVER".  
IT MEANS GROUP THY IS INTERESTING! AND SOMETIMES  
DIFFICULT.

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### ④ EDGE AND WORD METRICS

DEF: SUPPOSE  $\Gamma$  IS A CONNECTED GRAPH. DEFINE

$d_\Gamma : V(\Gamma) \times V(\Gamma) \longrightarrow \mathbb{N}$  BY

$$d_\Gamma(u, v) = \min \{ n \mid \text{THERE IS AN EDGE PATH } \gamma : [0, n] \rightarrow \Gamma \text{ WITH } \gamma(0) = u, \gamma(n) = v \}$$

THIS IS THE EDGE METRIC ON  $\Gamma$ . WE EXTEND TO THE INTERIORS OF EDGES BY GIVING THEM LENGTH ONE.

EXERCISE:  $d_\Gamma$  IS A METRIC.

DEF: SUPPOSE  $\Gamma = \Gamma(G, S)$ . WE DEFINE  $d_S(g, h) = d_\Gamma(g, h)$  AND  $|g|_S = d_S(1, g)$ . THESE ARE THE WORD METRIC AND WORD NORM ON  $G$  ASSOCIATED TO  $S$

LEMMA: THE LEFT ACTION OF  $G$  ON  $\Gamma(G, S)$  IS VIA ISOMETRIES OF  $d_S$ . THAT IS: FOR ALL  $f, g, h \in G$

$$d_S(fg, fh) = d_S(g, h).$$

PROOF: EXERCISE. □

EXERCISES: ①  $d_S(g, h) = |g^{-1}h|_S$

$$② |g|_S = |g^{-1}|_S$$

$$③ |g \cdot h|_S \leq |g|_S + |h|_S$$

EXERCISE: WITH THE USUAL GEN SET, THE WORD METRIC ON  $\mathbb{Z}^m$  IS THE RESTRICTION OF THE  $L^1$  METRIC ON  $\mathbb{R}^m$ :  $d_1(x, y) = \sum_{i=1}^m |x_i - y_i|$ .

[ MORE GENERALLY: CONSIDER  $H_3(\mathbb{Z}) < H_3(\mathbb{R})$  OR  $SL_d(\mathbb{Z}) < SL_d(\mathbb{R})$  OR ANY NICE LATTICE IN A REAL LIE GROUP OR THE INTEGER POINTS OF AN ALGEBRAIC GROUP SAY  $SO_{\mathbb{R}}(1, n) \dots$  ]