2025-01-21 SAUL SCHLEIMER MA444 LECTURE 7

① UNIV. PROP. of GROUP PRESENTATIONS: SUPPOSE $G = \langle S \mid R \rangle$ IS A PRESENTATION, AND $\bar{\varphi}: F(S) \to G$ IS THE INDUCED HOMOMORPHISM. SUPPOSE H IS A GROUP AND $\psi: S \to H$ IS ANY FUNCTION. SUPPOSE $R \subset \mathrm{KER}(\bar{\psi}: F(S) \to H)$. THEN THERE EXISTS A UNIQUE HOMOMORPHISM $\Psi: G \to H$ SO THAT $\bar{\psi} = \Psi \circ \bar{\varphi}$.

DIAGRAM:

$$\begin{tikzcd}
 & & G \arrow[d, "\Psi"] \\
S \arrow[r] \arrow[urr, bend left, "\varphi"] \arrow[drr, bend right, "\psi"'] & F(S) \arrow[ur, "\bar{\varphi}"] \arrow[dr, "\bar{\psi}"'] \arrow[r, phantom, "\circ"] & {} \\
 & & H
\end{tikzcd}$$

THAT IS: "G IS THE BIGGEST GROUP SATISFYING THE RELATORS R".

EXERCISE: PROVE THIS.

---

② WORD PROBLEM: [DEHN 1911]. NOW TAKE S FINITE. ALL of THE CONSTRUCTIONS of CAYLEY GRAPHS RELY ON SOLVING THE:

WORD PROBLEM FOR (G,S): GIVEN $u, v \in (S \cup S^{-1})^*$ DECIDE IF $u =_G v$. [IF $\bar{\varphi} \circ r(u) = \bar{\varphi} \circ r(v)$]

THAT IS, FOR (G,S) FIXED FIND AN ALGORITHM THAT TAKES AS INPUT u AND v AND OUTPUTS

"YES" IFF $u =_G v$

"NO" IFF $u \neq_G v$.

EQUIVALENTLY INPUT IS $w \in (S \cup S^{-1})^*$ AND DECIDE IF $w =_G 1_G$.

EXAMPLES: ① $(F(S), S)$: REDUCE u, v TO OBTAIN $r(u), r(v)$ AND CHECK EQUALITY AS STRINGS.

② $(\mathbb{Z}^2, \{(1,0), (0,1)\})$: ADD.

EXERCISE: GIVEN ALGORITHMS FOR $WP(G,S)$, $WP(H,T)$ FIND AN ALGORITHM FOR $WP(G \times H, S \cup T)$.

CHALLANGE: SOLVE WORD PROBLEM FOR $BS(1,2)$.

EXERCISE: SUPPOSE S, S' ARE GEN SETS FOR G. THEN $WP(G,S)$ SOLVABLE IFF $WP(G,S')$ SOLVABLE.

③ UNDECIDABILITY

THM [NOVIKOV, 1955] THERE IS A FIN. PRES GROUP $G = \langle S \mid R \rangle$ SO THAT $WP(G,S)$ IS UNDECIDABLE.

COR: FOR THIS GROUP WE CANNOT BUILD $\Gamma(G,S)$.

THEOREM [ADIAN 1957, RABIN 1958] THERE IS NO ALGORITHM THAT, GIVEN FIN. PRES $G = \langle S \mid R \rangle$, DECIDES IF

G IS
- TRIVIAL
- FINITE
- COMMUTATIVE
- NILPOTENT
- SOLVABLE
- TORSION-FREE
- RES. FINITE
- SOLVABLE WORD PROB
- SIMPLE
- AUTOMATIC ...

OR

[SEE MILLER 1992]

NOTE: THIS DOES NOT MEAN GROUP THY IS "OVER". IT MEANS GROUP THY IS INTERESTING! AND SOMETIMES DIFFICULT.

---

④ EDGE AND WORD METRICS

DEF: SUPPOSE Γ IS A CONNECTED GRAPH. DEFINE

$$d_\Gamma : V(\Gamma) \times V(\Gamma) \longrightarrow \mathbb{N} \text{ BY}$$

$$d_\Gamma(u,v) = \min \left\{ n \;\middle|\; \begin{array}{l} \text{THERE IS AN EDGE PATH } \gamma : [0,n] \to \Gamma \\ \text{WITH } \gamma(0) = u,\ \gamma(n) = v \end{array} \right\}$$

THIS IS THE <u>EDGE METRIC</u> ON Γ. WE EXTEND TO THE INTERIORS OF EDGES BY GIVING THEM LENGTH ONE.

<u>EXERCISE</u>: d_Γ IS A METRIC.

<u>DEF</u>: SUPPOSE $\Gamma = \Gamma(G,S)$. WE DEFINE $d_S(g,h) = d_\Gamma(g,h)$ AND $|g|_S = d_S(1_G, g)$. THESE ARE THE <u>WORD METRIC</u> AND <u>WORD NORM</u> ON G ASSOCIATED TO S

<u>LEMMA</u>: THE LEFT ACTION OF G ON $\Gamma(G,S)$ IS VIA ISOMETRIES OF d_S. THAT IS: FOR ALL $f, g, h \in G$

$$d_S(fg, fh) = d_S(g,h).$$

<u>PROOF</u>: EXERCISE. $\square$

<u>EXERCISES</u>: ① $d_S(g,h) = |g^{-1}h|_S$

② $|g|_S = |g^{-1}|_S$

③ $|g \cdot h|_S \leq |g|_S + |h|_S$

<u>EXERCISE</u>: WITH THE USUAL GEN SET, THE WORD METRIC ON $\mathbb{Z}^m$ IS THE RESTRICTION OF THE L^1 METRIC ON $\mathbb{R}^m$: $d_1(x,y) = \sum_{i=1}^{m} |x_i - y_i|$.

[MORE GENERALLY: CONSIDER $H_3(\mathbb{Z}) < H_3(\mathbb{R})$ OR $SL_d(\mathbb{Z}) < SL_d(\mathbb{R})$ OR ANY NICE LATTICE IN A REAL LIE GROUP OR THE INTEGER POINTS OF AN ALGEBRAIC GROUP SAY $SO_{\mathbb{R}}(1,n)$ ...]