2025-01-16 LECTURE 6 MA4H4 SAUL SCHLEIMER

① **CAYLEY GRAPHS**: G A GROUP, $S \subset G$ A GEN SET. DEFINE THE CAYLEY GRAPH $\Gamma = \Gamma(G,S)$ WITH

VERTICES $V(\Gamma) = G$

EDGES $E(\Gamma) = \{(g, gs) \mid g \in G, s \in S\}$

LEMMA [ACTION] G ACTS (ON THE LEFT) ON $\Gamma(G,S)$ VIA GRAPH AUTOMORPHISMS.

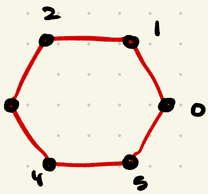
PROOF: $h \cdot g = hg$ AND $h \cdot (g, gs) = (hg, hgs) \in E(\Gamma)$

ALSO, THESE ARE BIJECTIONS (CONSIDER ACTION of h^{-1}). $\square$

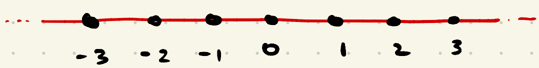
THE ACTION IS VERTEX TRANSITIVE BUT TYPICALLY NOT EDGE TRANSITIVE. ASSUMING NO $s \in S$ IS ORDER TWO, $G \backslash \Gamma(G,S)$ IS A ROSE WITH EDGES LABELLED BY ELTS of S.

EXAMPLES:

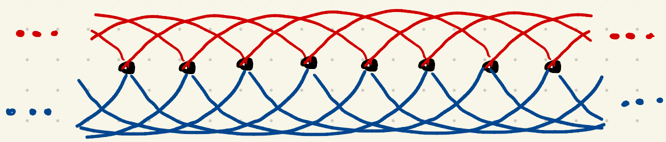
$\Gamma(\mathbb{Z}/6\mathbb{Z}, \{1\})$



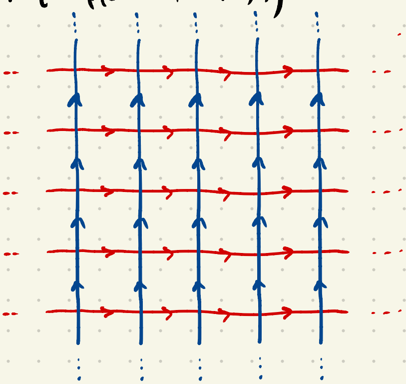
$\Gamma(\mathbb{Z}, \{1\})$



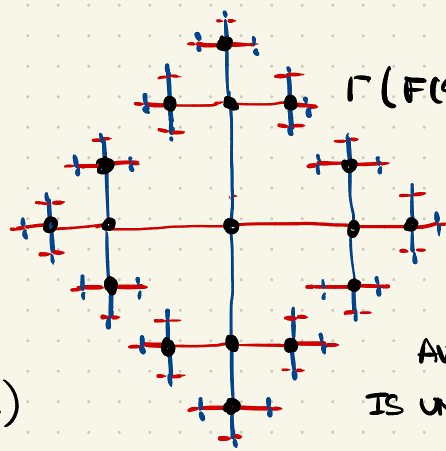
$\Gamma(\mathbb{Z}, \{3,4\})$



$\Gamma(\mathbb{Z}^2, \{(1,0),(0,1)\})$



$\Gamma(F(S), S)$

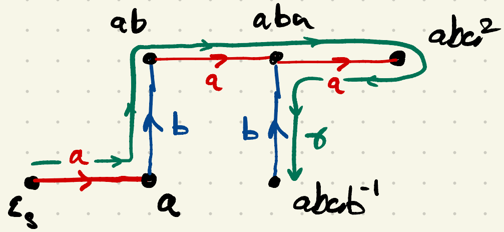


NOTE: IF $s^2 = 1$ AND $h = gs$ WE MERGE (g,gs) AND (h,hs)

NOTE: $\mathrm{Aut}(\Gamma(F(S),S))$ IS UNCOUNTABLE

AN EDGE PATH $\gamma : [0,n] \to \Gamma(G,S)$ IS A CTS PATH WHICH IS A CONCAT. of EDGES. γ IS AN EDGE LOOP IF $\gamma(n) = \gamma(0)$. THE LABEL of γ IS THE WORD $w \in (S \cup S^{-1})^*$ WHERE w_i IS THE LABEL of THE EDGE $\gamma|[i,i+1]$.

EXAMPLE $\gamma : [0,6] \to \Gamma(F(S),S)$



THE LABEL of γ IS $abaaa^{-1}b^{-1}$ [WHICH REDUCES TO $abab^{-1}$]

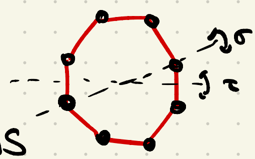
NOTE THAT $g \cdot \gamma$ AND γ HAVE THE SAME LABEL.

CONVERSELY: GIVEN $g \in G$ AND $w \in (S \cup S^{-1})^*$ WE HAVE A (UNIQUE) PATH $\gamma : [0, |w|] \to \Gamma(G,S)$ WITH $\gamma(0) = g$ AND LABEL w.

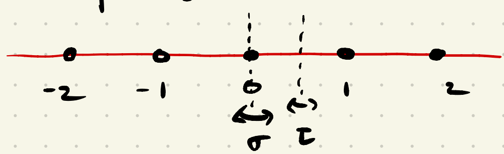
② MANY EXERCISES

DRAW THE CAYLEY GRAPHS $\Gamma(G,S)$ FOR THE FOLLOWING

(i) $G = D_{2n}$, $S = \{\sigma, \tau\}$ A PAIR of ADJACENT REFLECTIONS



(ii) $G = D_\infty$, $S = \{\sigma, \tau\}$ A PAIR of ADJ. REFLECTIONS



(iii) $G = SL(2,\mathbb{Z})$, $S = \{L, R\}$ WITH $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

(iv) $G = SL(2,\mathbb{Z})$, $S = \{A, B\}$ WITH $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

[IN PARTICULAR, CHECK A, B GENERATE]

(v) $G = SL(2,\mathbb{Z})$, $S = \{A, R\}$.

(vi) $G = BS(1,2) = \langle a, b \mid aba^{-1} = b^2 \rangle$

EXERCISE: SUPPOSE Γ IS A NON-EMPTY CONNECTED GRAPH.

THE FOLLOWING ARE EQUIVALENT:

(i) EVERY EDGE SEPARATES

(ii) Γ HAS NO EMBEDDED EDGE LOOP

(iii) ANY $x, y \in \Gamma$ ARE CONN. BY A UNIQUE EMBEDDED EDGE PATH.

(iv) Γ IS CONTRACTIBLE (iv') Γ DEF RETRACTS TO A POINT.

(v) $\pi_1(\Gamma, v) \cong 1$ (vi) Γ IS 0-HYPERBOLIC. (vii) MEDIAN SPACES.-

ADDITIONAL, IF Γ IS FINITE:

(a) $|V(\Gamma)| - |E(\Gamma)| = 1$

(b) Γ HAS A LEAF ℓ SO THAT $\Gamma - \ell$ IS A TREE (OR $\Gamma = \{pt\}$)

LEMMA: (1) $\Gamma(G,S)$ IS CONNECTED.

(2) $\Gamma(F(S),S)$ IS A TREE

(3) $\Gamma(F(S),S)$ IS A UNIV. COVER of $\Gamma(G,S)$.

PROOF: (1) BY HYPOTHESIS, $\langle S \rangle = G$ SO $F(S) \longrightarrow G$ SURJECTS.

SO EVERY $g \in G$ HAS SOME $w \in F(S)$ WITH $\varphi(w) = g$.

SO w GIVES A LABELLED PATH IN $\Gamma(G,S)$ FROM 1_G TO g.

(2) SUPPOSE $\gamma : [0,n] \longrightarrow \Gamma = \Gamma(F(S),S)$ IS AN EDGE LOOP. SUPPOSE $n > 0$. LET w BE THE LABEL of γ. SINCE Γ IS A CAYLEY GRAPH $w = \varepsilon_S$ AS ELEMENTS of $F(S)$. SO [BY REDUCED WORD THM] w IS NOT REDUCED. SO γ IS NOT AN EMBEDDING.

(3) $\bar{\varphi} : F(S) \to G$ INDUCES A COVERING AND $\pi_1(\Gamma(F(S),S)) \cong 1$. $\square$

PICTURE