2025-01-15 LECTURE 5 SAUL SCHLEIMER MA4H4

① PRESENTATIONS:

DEF: SUPPOSE S IS A SET. SUPPOSE $R \subset F(S)$ IS A SET. WE DEFINE THE GROUP $\langle S \mid R \rangle$ GEN BY S WITH RELATORS R BY $\langle S \mid R \rangle = F(S)/\langle\langle R \rangle\rangle$.

STANDARD ABUSE of LANGUAGE: $\langle S \mid R \rangle$ REMEMBERS S AND R.

LEMMA: EVERY GROUP HAS A PRESENTATION.

PROOF: TAKE $S = G$, $R = \ker(i)$. DEDUCE $G \cong \langle S \mid R \rangle$ FROM FIRST ISOMORPHISM THEOREM. $\square$

NOTE: PRESENTATIONS ARE NEVER UNIQUE:

$\langle S \mid R \rangle \cong \langle S \cup \{t\} \mid R \cup \{t\} \rangle$ WHERE t IS A NEW LETTER.

CONVENTION: WE OMIT SET BRACKETS WHEN POSSIBLE:

EX: $\mathbb{1} \cong \langle \mid \rangle$, $\mathbb{1} \cong \langle a \mid a \rangle$

$\mathbb{Z} \cong \langle a \mid \rangle$, $\mathbb{Z} \cong \langle a, b \mid b \rangle$, $\mathbb{Z}/n\mathbb{Z} \cong U_n \cong \langle a \mid a^n \rangle$

$\mathbb{Z}^2 \cong \langle a, b \mid [a,b] \rangle$ RECALL $[a,b] = aba^{-1}b^{-1}$.

$\mathbb{Z}^3 \cong \langle a, b, c \mid [a,b], [b,c], [c,a] \rangle$.

$\pi_1($  $) \cong \langle a, b, c, d \mid abcd\,a^{-1}b^{-1}c^{-1}d^{-1} \rangle$

EXERCISES

EX: $F(S) \cong \langle S \mid \rangle$, $\mathbb{1} = \langle S \mid S \rangle$

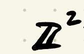
DEF: G IS FINITELY PRESENTED IF THERE IS S, R SO THAT $G \cong \langle S \mid R \rangle$ AND $|S|, |R| < \infty$.

EXAMPLES: ① FINITE GROUPS [USE $S = G$, $R =$ MULTI TABLE]

② $\mathbb{Z}^m$, $F(S)$, π_1(FINITE CW-COMPLEX), ...

② RELATIONS: SUPPOSE $uv^{-1} \in R$ IS A RELATOR. THEN WE MAY, INSTEAD, WRITE THE RELATION $u = v$. [THIS DOES NOT MEAN

$u=v$ IN S; RATHER IT MEANS THEIR IMAGES ARE EQUAL IN $\langle S \mid R\rangle$.]

SO: $\mathbb{Z}^2 \cong \langle a,b \mid ab=ba\rangle$, $\pi_1($  $) = \langle a,b,c,d \mid abcd=dcba\rangle$

DEFINE: $BS(p,q) = \langle a,b \mid ab^p = b^q a\rangle$ [BAUMSLAG-SOLITAR GROUP]

NOTE: $BS(1,1) \cong \mathbb{Z}^2$.

③ PRESENTATIONS ARE MYSTERIOUS

DEFINE $H_k = \langle x_0, x_1, \dots, x_{k-1} \mid x_i x_{i+1} x_i^{-1} = x_{i+1}^2$ FOR $i=0,\dots,k-1$ MOD $k\rangle$

THEOREM [HIGMAN, 1951] $H_k \cong 1$ FOR $k=0,1,2,3$. FOR $k \geq 4$, H_k IS INFINITE AND HAS NO FINITE QUOTIENTS.

EXAMPLE: $H_2 \cong \langle a,b \mid aba^{-1}=b^2,\ bab^{-1}=a^2\rangle$

SO $aba^{-1}b^{-1}=b$, $bab^{-1}a^{-1}=a$. WE INVERT THE LATTER TO GET $aba^{-1}b^{-1}=a^{-1}$. SO $a^{-1}=b$. SO a AND b COMMUTE! SO $a=b=1$ AND WE ARE DONE. □

EXERCISE: $\langle a,b \mid ab^3a^{-1}=b^2,\ ba^3b^{-1}=a^2\rangle \cong 1$

[TAKEN FROM BURNS-MACEDONSKA, 1993]

④ NOT FIN. PRES. GROUPS.

SINCE FIN PRESENTED GROUPS ARE FIN. GEN WE HAVE

(*) $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ ARE NOT FIN PRES

HERE ARE TWO FIN GEN GROUPS WHICH ARE NOT FIN. PRES:

(1) GRIGORCHUK'S GROUP of INTERMED GROWTH.

[LYSENOK 1985] SET $S=\{a,b,c,d\}$ AND DEFINE $\phi: S^* \to S^*$ BY $\phi(a)=aca$, $\phi(b)=d$, $\phi(c)=b$, $\phi(d)=c$, AND EXTEND RECURSIVELY.

DEFINE $w_0=ad$, $w_{n+1}=\phi(w_n)$.

DEFINE $G = \left\{ a, b, c, d \;\middle|\; \begin{array}{l} a^2 = b^2 = c^2 = d^2 = bcd = 1 \\ w_n^4 = (w_n w_{n+1})^4 = 1, \; n \geq 0 \end{array} \right\}$

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② THE LAMPLIGHTER GROUP

$$L = \langle a, t \mid a^2, \; [t^m a t^{-m}, t^n a t^{-n}] \quad m, n \in \mathbb{Z} \rangle$$

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⑤ CAYLEY GRAPHS:

DEF: SUPPOSE G IS A GROUP. SUPPOSE $S \subset G$ GENERATES.
WE DEFINE THE CAYLEY GRAPH [DEHNSCHE GRUPPENBILDER]
$\Gamma = \Gamma(G, S)$ TO HAVE: VERTEX SET: $V(\Gamma) = G$

EDGE SET: $E(\Gamma) = \{ (g, gs) \mid g \in G, s \in S \}$

EXAMPLE: FOR $\mathbb{Z}/6\mathbb{Z}$ GEN BY 1: