LECTURE 4 MA4H4 2025-01-14 SAUL SCHLEIMER

ADMIN: ALL ADMIN DETAILS ON WEBPAGES

ALSO LECTURE NOTES AND EXECISES

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① UNIVERSAL PROPERTY of FREE GROUPS.

THEOREM: SUPPOSE S IS A SET, G A GROUP. SUPPOSE $\varphi : S \to G$ IS ANY FUNCTION. THEN THERE IS A UNIQUE HOMOMORPHISM $\bar{\varphi} : F(S) \longrightarrow G$ EXTENDING φ.

DIAGRAM

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\iota"'] & G \\ F(S) \arrow[ur, dashed, "\bar{\varphi}"'] & \end{tikzcd}$$

PROOF: WE DEFINE $\bar{\varphi}$ BY RECURSION:

(*) $\bar{\varphi}(\varepsilon_S) = 1_G$

(*) $\bar{\varphi}(s) = \varphi(s)$, $\bar{\varphi}(s^{-1}) = (\varphi(s))^{-1}$

(*) $\bar{\varphi}(w * s) = \bar{\varphi}(w) \cdot \bar{\varphi}(s)$

FOR $w * s \in F(S)$; $s \in S \cup S^{-1}$

CLAIM 1: $\bar{\varphi}(s * w) = \bar{\varphi}(s) \cdot \bar{\varphi}(w)$

CLAIM 2: $\bar{\varphi}(w^{-1}) = (\bar{\varphi}(w))^{-1}$

CLAIM 3: SUPPOSE $u \cdot v = u * v$ IN $F(S)$. THEN $\bar{\varphi}(u \cdot v) = \bar{\varphi}(u) \cdot \bar{\varphi}(v)$.

$\}$ EXERCISES.

CLAIM: SUPPOSE $u, v \in F(S)$. THEN $\bar{\varphi}(u \cdot v) = \bar{\varphi}(u) \cdot \bar{\varphi}(v)$.

PROOF: INDUCT ON $|w|$ WITH w GIVEN BY CANCELLATION LEMMA. IF $|w| = 0$ THEN DONE BY CLAIM.

OTHERWISE

$$\begin{aligned}
\bar{\varphi}(u \cdot v) &= \bar{\varphi}(u' * v') && \text{CANCEL. LEM.} \\
&= \bar{\varphi}(u') \bar{\varphi}(v') && \text{CLAIM 3} \\
&= \bar{\varphi}(u') \bar{\varphi}(w) (\bar{\varphi}(w))^{-1} \bar{\varphi}(v') && G \text{ GROUP} \\
&= \bar{\varphi}(u') \bar{\varphi}(w) \bar{\varphi}(w^{-1}) \bar{\varphi}(v') && \text{CLAIM 2} \\
&= \bar{\varphi}(u' w) \bar{\varphi}(w^{-1} \cdot v') && \text{CLAIM 3} \\
&= \bar{\varphi}(u) \bar{\varphi}(v) && \text{CANCEL LEM.}
\end{aligned}$$

THUS $\bar{\varphi}$ IS A HOMOMORPHISM.

UNIQUENESS. SUPPOSE ϕ, ψ EXTEND φ. THEN

$$
\begin{aligned}
\phi(w * s) &= \phi(w)\,\phi(s) && \phi \text{ HOMOMOR.} \\
&= \psi(w)\,\phi(s) && \text{INDUCTION} \\
&= \psi(w)\,\varphi(s) && \text{EXTENSION} \\
&= \psi(w)\,\psi(s) && \text{EXTENSION} \\
&= \psi(w * s) && \psi \text{ HOMOMOR.}
\end{aligned}
$$

□

COROLLARY: A BIJECTION $\varphi : S \to T$ INDUCES AN ISOMORPHISM $\bar{\varphi} : F(S) \to F(T)$. □

CHALLENGE: PROVE A CONVERSE: IF $F(S) \cong F(T)$ THEN $|S| = |T|$.

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② **GENERATORS**: REVIEW DEFS OF SUBGROUP, NORMAL SUBGROUP.

DEF: SUPPOSE (*) $S \subset G$. LET $\langle S \rangle < G$ BE THE IMAGE OF THE INDUCED HOMOMORPHISM $\bar{\iota}_S : F(S) \to G$. WE CALL $\langle S \rangle$ THE SUBGROUP GENERATED BY S.

(*) SUPER TECHNICAL POINT. IT IS BETTER TO USE $\varphi : S \to G$ INSTEAD OF $S \subset G$. WE THEN WOULD TALK ABOUT "THE SUBGROUP $\langle \varphi \rangle$", NAMELY $\langle \varphi \rangle = \text{IMAGE}(\bar{\varphi})$. THIS IS BETTER THAN $\langle S \rangle$ FOR THE SAME REASON LISTS ARE BETTER THAN SETS. BUT NOBODY TALKS THIS WAY. SO. NVM.

LEMMA: $\langle S \rangle = \bigcap_{S \subset H < G} H$ **PROOF**: EXERCISE □

EXERCISE: IF S FINITE THEN $\langle S \rangle$ IS COUNTABLE.

DEF: SUPPOSE THERE IS SOME $S \subset G$ FINITE SO THAT $G = \langle S \rangle$. THEN WE SAY G IS FIN. GENERATED.

EXAMPLES:
- FIN. GPS ARE FIN GEN.
- $\mathbb{Z}^n$, $F(S)$ [IF S FINITE]
- $SL(d, \mathbb{Z})$, $H_3(\mathbb{Z})$

EXERCISE: PROVE $\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}$ GENERATE $SL(2,\mathbb{Z})$.

EXERCISE. PROVE $H_3(\mathbb{Z})$ IS FIN. GEN.

NOTE: $\mathbb{R}, \mathbb{C}, SL(d,\mathbb{R})$ -- LIE GROUPS NOT FIN. GEN.

EXERCISE: $\mathbb{Q}$ NOT FIN. GEN.

DEF: THE RANK of G IS THE MINIMAL CARDINALITY AMONG GEN SETS FOR G.

EXAMPLES: $\text{RANK}(\mathbb{Z}^m) = m$, $\text{RANK}(F(S)) = |S|$.

THEOREM: SUPPOSE G IS A FINITE SIMPLE GROUP. THEN $\text{RANK}(G) \leq 2$. ["PROOF" IS THOUSANDS of PAGES...]

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③ NORMAL CLOSURES

DEF: SUPPOSE G IS A GROUP. SUPPOSE $R \subset G$.

WE DEFINE $\langle\langle R \rangle\rangle$ TO BE THE NORMAL CLOSURE of R IN G: THAT IS

* $1_G \in \langle\langle R \rangle\rangle$
* $w g r g^{-1} \in \langle\langle R \rangle\rangle$

FOR ALL $w \in \langle\langle R \rangle\rangle$, $g \in G$, $r \in R \cup R^{-1}$.

LEMMA: $\langle\langle R \rangle\rangle = \bigcap_{R \subset N \triangleleft G} N$ $\quad$ [AND $\langle\langle R \rangle\rangle \triangleleft G$]

PROOF EXERCISE $\square$

IF $\langle\langle R \rangle\rangle = G$ WE SAY R NORMALLY GENERATES G

EXERCISE: $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ NORMALLY GENERATES $SL(2,\mathbb{Z})$.

EXERCISE: ANY ONE ELEMENTARY MATRIX GENERATES $SL(d,\mathbb{Z})$.

EXERCISE: $\langle\langle a \rangle\rangle \neq F(\{a,b\})$.