2025-01-09 LECTURE 3 SAUL SCHLEIMER MA4H4

TIMING of SUPPORT CLASS

PICK A TIME IN: TUESDAY AFTER 12PM
THURSDAY BEFORE 1PM
THURSDAY AFTER 3 PM
FRIDAY AFTER 1PM

THUR 12-1?

---

① DIAMONDS

THEOREM [REDUCED WORD] FOR ANY $w \in (S \cup S^{-1})^*$ THE CLASS $[w]$ CONTAINS EXACTLY ONE REDUCED WORD.

PROOF: EXISTENCE LAST TIME. FOR UNIQUENESS WE NEED:

DIAMOND LEMMA. [ALSO CALLED PEAK REDUCTION LEMMA] SUPPOSE $w', w'' \in (S \cup S^{-1})^*$ ARE REDUCTIONS of $w \in (S \cup S^{-1})^*$. THEN EITHER $w' = w''$ OR THEY HAVE A COMMON REDUCTION $w'' \in (S \cup S^{-1})^*$.

PICTURE

$$\begin{tikzcd}
w \arrow[d, dash] & \text{OR} & & w \arrow[dl, dash] \arrow[dr, dash] & \\
w' & & w' \arrow[dr, dash] & & w'' \arrow[dl, dash] \\
& & & w''' &
\end{tikzcd}$$

PROOF of LEMMA: SUPPOSE THAT THE REDUCTIONS OCCUR AT THE SUBWORDS $w_i w_{i+1}$ AND $w_j w_{j+1}$ of w. WE MAY ASSUME THAT $i \leq j$, GIVING THREE CASES

A) $i = j$ B) $j = i+1$ C) $j \geq i+2$

A) HERE $w = utt^{-1}v$ AND $w' = w'' = uv$.

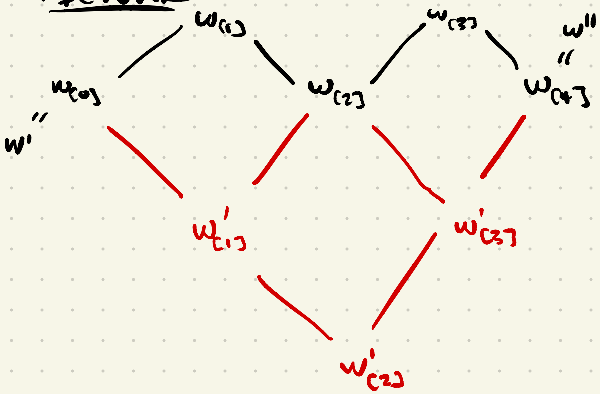
B) HERE $w = utt^{-1}tv$ AND $w' = w'' = utv$.

C) HERE $w = uss^{-1}u'tt^{-1}u''$ AND $\left.\begin{matrix} w' = uu'tt^{-1}u'' \\ w'' = uss^{-1}u'u'' \end{matrix}\right\} w''' = uu'u''$.

$\square$
(LEMMA)

SUPPOSE NOW THAT $w', w'' \in [w]$. BY DEFINITION THERE ARE WORDS $(w_{[i]})_{i=0}^{N}$ IN $[w]$ SO THAT $w' = w_{[0]}, w'' = w_{[N]}$, AND $w_{[i+1]}$ IS AN EXP/RED of $w_{[i]}$.

PICTURE



THE NEW PATHS PRODUCED BY THE DIAMOND LEMMA REDUCE $\sum_i |w_{[i]}|$ SO EVENTUALLY TERMINATE IN A NEW PATH WITH ALL REDUCTIONS BEFORE ALL EXPANSIONS.

THUS IF w', w'' ARE REDUCED THE NEW PATH HAS LENGTH ZERO AND $w' = w''$, AS DESIRED. □ (THEOREM)

② THE FREE GROUP

DEF: SET $F(S) = \{ w \in (S \cup S^{-1})^* \mid w \text{ REDUCED} \}$.

DEF: $r : (S \cup S^{-1})^* \longrightarrow F(S)$ TAKES w TO THE REDUCED WORD IN $[w]$. [WELL-DEF BY THEOREM].

DEF $\cdot : F(S) \times F(S) \longrightarrow F(S)$ BY $(u, v) \longmapsto r(u * v)$.

THEOREM: $(F(S), \cdot)$ IS A GROUP WITH IDENTITY ε_S AND INVERSE $w \longmapsto w^{-1}$

PROOF: FIX $u, v, w \in F(S)$ REDUCED WORDS.

⊛ $\varepsilon_S \cdot u = r(\varepsilon_S * u) = r(u) = u$ AS u REDUCED. $u \cdot \varepsilon_S$ SIMILAR.

⊛ $\varepsilon_S \cdot \varepsilon_S^{-1} = \varepsilon_S \cdot \varepsilon_S = r(\varepsilon_S * \varepsilon_S) = r(\varepsilon_S) = \varepsilon_S$. NOW INDUCT

$$\begin{aligned} (w*s)\cdot(w*s)^{-1} &= (w*s)\cdot(s^{-1}*w^{-1}) && \text{DEF INV} \\ &= r(w*s*s^{-1}*w^{-1}) && \text{DEF } \cdot \\ &= r(w*w^{-1}) && \text{THEOREM} \end{aligned}$$

$= w \cdot w^{-1}$ DEF $\cdot$

$= \varepsilon_S$ INDUCTION

$\circledast$ $u \cdot (v \cdot w) = r(u * r(v * w))$ DEF r

$= r(u * v * w)$ THEOREM

$= r(r(u * v) * w)$ THEOREM

$= (u \cdot v) \cdot w$ DEF r □

<u>DEF</u>: WE CALL $F(S)$ THE <u>FREE GROUP</u> <u>GEN. BY</u> S.

③ <u>CANCELLATION LEMMA</u>: SUPPOSE $u, v \in F(S)$. THEN THERE ARE <u>UNIQUE</u> $u', v', w \in F(S)$ SO THAT

$$\left.\begin{array}{l} u = u' \cdot w \\ v = w^{-1} \cdot v' \end{array}\right\} \text{ AND } u \cdot v = u' \cdot v' = u' * v'.$$

<u>PROOF</u>: EXERCISE [HINT: INDUCT ON $|u|$, CASES...] □

THIS IS USEFUL, ...

<u>EXERCISES</u>:

(1) PROVE $F(S)$ COMMUTATIVE IFF $|S| \leq 1$

(2) PROVE $F(S)$ HAS NO NON-TRIVIAL TORSION.

(3) CHALLENGE: PROVE THAT ALL SUBGROUPS of $F(S)$ ARE ISOMORPHIC TO FREE GROUPS. [START WITH FIN. GEN SUBGROUPS.]

<u>EXERCISES</u>: CALL $\mathbb{Z}^m$ THE <u>FREE COMM. GROUP</u> of RANK m.

① PROVE $\mathbb{Z}^m$ HAS NO NON TRIV. TORSION.

② PROVE $\mathbb{Z}^m \cong \mathbb{Z}^n$ IFF $m = n$ [JUSTIFYING NAME "RANK"]

(3) PROVE $\mathbb{Z}^m$ SURJ. $\mathbb{Z}^n$ IFF $m \geq n$

(4) PROVE G INJECTS $\mathbb{Z}^n$ IFF $G \cong \mathbb{Z}^m$ FOR SOME $m \leq n$.