2025-01-08 MA4H4 LECTURE 2 SAUL SCHLEIMER | 7 STUDENTS

① GROUP ACTIONS

DEF: SUPPOSE G IS A GROUP. SUPPOSE X IS A MATHEMATICAL OBJECT. AN ACTION of G ON X IS A HOMOMORPHISM $\rho : G \to \mathrm{AUT}(X)$. IN TERMS of THE UNDERLYING SET, WE OBTAIN $A_\rho : G \times X \to X$ WITH

$$A_\rho(g, x) = \rho_g(x).$$

EXERCISE: THE USUAL AXIOMS FOR AN ACTION ON A SET FOLLOW

- $e_G \cdot x = x$ FOR ALL $x \in X$
- $g \cdot (h \cdot x) = (gh) \cdot x$ FOR ALL $g, h \in G$, $x \in X$.

FOR US THERE ARE TWO IMPORTANT CASES

- X A METRIC SPACE
- X A GRAPH

EXERCISE: FIND ALL ACTIONS of $\mathbb{Z}$ ON

- THE METRIC SPACE $\mathbb{R}$, $d_{\mathbb{R}}(x, y) = |x - y|$
- THE VECTOR SPACE $\mathbb{R}$ (OVER $\mathbb{R}$!)
- THE GRAPH $\mathbb{R}$ 

[THIS IS ALMOST THE SAME AS COMPUTING $\mathrm{AUT}(X)$]

[WHAT IS $\mathrm{AUT}(\mathbb{R})$ WHEN $\mathbb{R}$ IS A FIELD, TOP SPACE, SET, SMOOTH MANIFOLD, ...]

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② WORDS:

WE NOW START OUR TREATMENT of FREE GROUPS.

DEF: FIX S A SET. A WORD OVER S of LENGTH $n \in \mathbb{N}$ IS A FUNCTION $w : \{0, 1, \ldots, n-1\} \to S$.

WE OFTEN WRITE w AS A LIST WITHOUT COMMAS.

EXAMPLE: $S=\{0,1\}$. WORDS OVER S ARE BINARY STRINGS.
THE FIRST FEW ARE ε_S, 0, 1, 00, 01, 10, 11, 000, 001, ...

NOTATION:
- S^* IS THE SET of WORDS OVER S.
- $|w|$ IS THE LENGTH of w
- ε_S IS THE ~~EMPTY~~ WORD OVER S, WITH $|w|=0$.
- $w_i = w(i)$ IS THE i^{th} LETTER of w
- FOR $u,v \in S^*$ LET $u*v = u_0 \cdots u_{k-1} v_0 \cdots v_{\ell-1}$ BE THEIR CONCATENATION

NOTE: $|u*v| = |u|+|v|$.

EX: GIVE CAREFUL DEFS of CONCATENATION, SUBWORD PREFIX AND SUFFIX.

EXERCISE: SET $S=\{0,1\}$.
- COUNT WORDS IN S^* of LENGTH n
- " " " " " " $\leq n$
- " " " " " " n THAT DO NOT HAVE 11 AS A SUBWORD.
- -- -- " 111 " " " [HARDER]

LEMMA: SUPPOSE S IS A SET. FOR ALL $u,v,w \in S^*$:
- $\varepsilon_S * u = u * \varepsilon_S = u$
- $(u*v)*w = u*(v*w)$.

SINCE $|u*v| = |u|*|v|$, ONLY ε_S HAS AN INVERSE (ITSELF).

DEF: FOR $n \in \mathbb{N}$ WE DEFINE u^n RECURSIVELY:
- $u^0 = \varepsilon_S$ AND
- $u^{n+1} = u^n * u$.

EXERCISE: SUPPOSE $u, v \in S^*$. SUPPOSE $u * v = v * u$.
THEN: THERE IS SOME $w \in S^*$, $p, q \in \mathbb{N}$ SO THAT

$$u = w^p \qquad \text{AND} \qquad v = w^q$$

LEMMA: WE CAN OBTAIN S^* VIA RECURSION: PLACE ε_S IN S^* AND, FOR ALL $w \in S^*, s \in S$, PLACE $w * s$ IN S^*. $\square$

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③ **INVERSES**: SUPPOSE S IS A SET. SUPPOSE S^{-1} IS A DISJOINT COPY OF S WITH $t^{-1} \in S^{-1}$ CORRESPONDING TO $t \in S$.
WE DEFINE AN INVOLUTION $\mathrm{INV}: S \cup S^{-1} \longrightarrow S \cup S^{-1}$ BY
$\mathrm{INV}(t) = t^{-1}$ AND $\mathrm{INV}(t^{-1}) = t$.
WE EXTEND INV TO ALL OF $(S \cup S^{-1})^*$ BY RECURSION:

$$\left.\begin{aligned} \mathrm{INV}(\varepsilon_S) &= \varepsilon_S \\ \mathrm{INV}(w * s) &= \mathrm{INV}(s) * \mathrm{INV}(w). \end{aligned}\right\} \text{SO } (w_0 w_1 \cdots w_{n-1})^{-1} = w_{n-1}^{-1} w_{n-2}^{-1} \cdots w_0^{-1}.$$

EXERCISE: $(w^{-1})^{-1} = w$ FOR ALL $w \in (S \cup S^{-1})^*$

④ **REDUCTIONS AND EXPANSIONS**

SUPPOSE $u, v \in (S \cup S^{-1})^*$. SUPPOSE $t \in S \cup S^{-1}$. SET

$w = u t t^{-1} v$ AND $w' = uv$.

WE CALL w AN *EXPANSION* OF w'
w' A *REDUCTION* OF w

CARTOON:

$$\begin{tikzcd} w = utt^{-1}v \arrow[d, shift right, "\text{REDUCE}"'] \\ uv = w' \arrow[u, shift right, "\text{EXPAND}"'] \end{tikzcd}$$

NOTE $|w'| = |w| - 2$. SO ANY SEQUENCE OF REDUCTIONS TERMINATES.
DEF: SAY $w \in (S \cup S^{-1})^*$ IS REDUCED IF IT HAS NO REDUCTIONS.

[THAT IS: $w_i \neq w_{i+1}^{-1}$ FOR $i = 0, 1, 2, \ldots n-2$]

WE USE REDUCTIONS/EXPANSIONS TO DEFINE AN

EQUIVALENCE RELATION ON $(S \cup S^{-1})^*$. LET $[w]$ BE THE EQUIV. CLASS of w.

NEXT TIME:

THEOREM: $[w]$ CONTAINS EXACTLY ONE REDUCED WORD.