2025-01-07 LECTURE 1 SAUL SCHLEIMER MA4H4

ADMIN: ON WEBPAGE.

GOAL: UNDERSTAND (SOME) GEOMETRIC ASPECTS OF FINITELY GENERATED GROUPS. SINCE SUCH GROUPS ARE ALWAYS COUNTABLE (EXERCISE!) THEY ARE "DISCRETE" SO THEIR "GEOMETRY" HAS TO BE CONSTRUCTED/UNDERSTOOD VIA THEIR "GEOMETRIC" ACTIONS ON "NICE" SPACES

FIRST WEEK IS A REVIEW.

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(1) GROUPS: A GROUP IS A SET G AND AN OPERATION $G \times G \to G$ SO THAT

*) THERE IS AN $e \in G$ SO THAT FOR ALL $g \in G$

- -) $e \cdot g = g \cdot e = g$
- =) THERE IS AN $h \in G$ SO THAT $g \cdot h = h \cdot g = e$

*) FOR ALL $f, g, h \in G$: $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

STANDARD EXERCISES:

- *) $e \in G$ UNIQUE, DENOTED e_G OR 0_G OR 1_G OR ε_G
- *) $h \in G$ UNIQUE GIVEN g, DENOTED g^{-1} OR $\bar{g}$ OR ..
- *) $(g^{-1})^{-1} = g$.

DEF: $(G, \cdot)$ IS COMMUTATIVE [ABELIAN] IF FOR ALL $g, h \in G$ $g \cdot h = h \cdot g$.

NOTATION: WE USE ADDITIVE OR MULT. NOTATION AS NEEDED: ADDITIVE ONLY FOR COMMUTATIVE GROUPS.

EXAMPLES:

*) $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ [ADDITIVE]

*) U_n, S^1, $\mathbb{Q}^*$, $\mathbb{R}^*$, $\mathbb{C}^*$ [MULTIPLICATIVE]

EXAMPLE *) $H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{Z} \right\}$

HEISENBURG GROUP

② MORPHISMS : WE SURPRESS THE GROUP OPERATION WHEN POSSIBLE.

DEF: SUPPOSE G, H ARE GROUPS. SUPPOSE $\varphi : G \to H$ IS A FUNCTION. SAY φ IS A HOMOMORPHISM IF, FOR ALL $g, h \in G$ WE HAVE

$$\varphi(gh) = \varphi(g) \cdot \varphi(h).$$

STANDARD: *) $\varphi(e_G) = e_H$

*) φ INJECTIVE IFF $\varphi^{-1}(e_H) = \{e_G\}$.

DEF: φ IS AN ISOMORPHISM IF THERE IS SOME HOMOMORPHISM $\psi : H \to G$ SO THAT $\varphi \circ \psi = Id_H$ AND $\psi \circ \varphi = Id_G$.

STANDARD: *) φ ISOMORPHISM IFF φ BIJECTIVE.

WE WRITE $G \cong H$ IF THEY ARE ISOMORPHIC [EXISTS AN ISOMORPHISM.]

EXERCISES: *) $\exp : (\mathbb{R}, +) \to (S^1, \cdot)$ IS HOMOMORPHISM.

*) $(\mathbb{Z}/n\mathbb{Z}, +) \cong (U_n, \cdot)$, $(\mathbb{R}, +) \cong (\mathbb{R}^*_{>0}, \cdot)$

③ NEW GROUPS FROM OLD : SUPPOSE G, H GROUPS

DEF: $AUT(G) = \{\varphi : G \to G \mid \varphi \text{ AN ISOM}\}$ IS THE AUTOMORPHISM GROUP.

DEF: $G \times H$ IS THE DIRECT PRODUCT : CARTESIAN PRODUCT of SETS AND POINT WISE OPERATION.

EXERCISE: *) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$.

*) PROVE $\mathbb{Z}^m \cong \mathbb{Z}^n$ IFF $m = n$. [TRY TO UNDERSTAND THE TOOLS YOU USE...]

DEF SUPPOSE $\varphi: G \to AUT(H)$ IS A HOMOMORPHISM.
DEFINE $H \rtimes_\varphi G$ THE SEMI-DIRECT PRODUCT BY TAKING CART. PRODUCT of SETS AND MULTIPLICATION

$$(h,g)\cdot(h',g') = (h\cdot\varphi(g)(h'), gg')$$

EXAMPLE: $AUT(\mathbb{Z}^2) \cong GL(2,\mathbb{Z})$. SET $\varphi: \mathbb{Z}^2 \to \mathbb{Z}^2$ TO BE $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: THAT IS $\varphi(x,y) = (x+y, y)$.

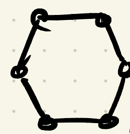
DEFINE $NIL = \mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}$.

EXERCISE: PROVE $NIL \cong H_3(\mathbb{Z})$

④ ALL of MATHEMATICS SUPPOSE X IS A MATHEMATICAL OBJECT - SET, SIMPLICIAL COMPLEX, TOP SPACE, VEC. SPACE, METRIC SPACE, FIELD, GROUP, ---

DEF $AUT(X)$ IS ITS GROUP of AUTOMORPHISMS.

EXAMPLE: C_n = THE CYCLE of LENGTH n

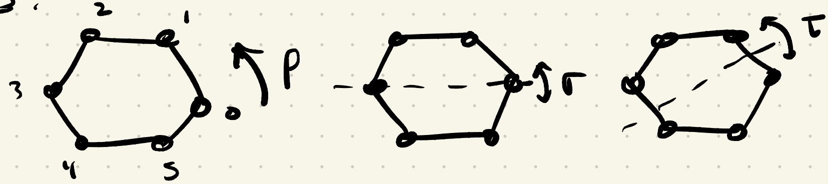


C_6

IF X IS A GRAPH, $\varphi \in AUT(X)$ PRESERVES THE VERTICES, EDGES, ADJACENCY.

$AUT(C_6)$ HAS ROTATIONS: SIX of THESE, INCLUDING ID.
REFLECTIONS: SIX of THESE, THREE DUAL TO VERTICES, THREE DUAL TO EDGES.

PICTURES:



NOTE: $\sigma^2 = \rho^6 = Id$ AND $\sigma\rho\sigma = \rho^{-1}$. ALSO $\tau \circ \sigma = \rho$.

DEF: $D_{2n} = AUT(C_n)$ THE DIHEDRAL GROUPS

EX: PROVE $D_{2n} \cong \mu_n \rtimes \mu_2$ WHERE μ_2 ACTS BY INVERSION.

A <u>SUBGROUP</u> $H \le G$ (a) CONTAINS e_G

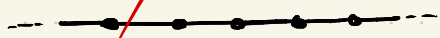
(b) IS CLOSED UNDER INVERSION

(c) IS " " MULTIPLICATION.

<u>EX</u>: FIND ALL SUBGROUPS of U_n, of D_{2n}.

<u>LEMMA</u>: AUT(G) PEMUTES THE SUBGROUPS of G. ∎

<u>EX</u>: SUPPOSE G HAS EXACTLY TWO SUBGROUPS: $\{e_G\}$ AND G. MUST AUT(G) BE TRIVIAL?

<u>EX</u>: LET L BE THE GRAPH 

SO $|L| \cong \mathbb{R}$ AS TOP SPACES. COMPUTE AUT(L).