Please let me (Saul) know if any of the problems are unclear or have typos.

Exercise 7.1. Suppose that U is a domain and that $z_0 \in U$. Suppose that $f, g \colon U \to \mathbb{C}$ are holomorphic. Suppose that $f(z_0) \neq 0$ and that g has a simple zero at z_0. Take $h = f/g$.

- Prove that $h \colon U \to \hat{\mathbb{C}}$ is meromorphic.

- Prove that h has a simple pole at z_0.

- Prove that $\mathrm{RES}(h, z_0) = f(z_0)/g'(z_0)$. $\diamond$

Exercise 7.2. Suppose that we are in the same setting as Exercise 7.1, except now g has a zero of order two at z_0. Prove that

$$\mathrm{RES}(h, z_0) = -\frac{2}{3} \cdot \frac{f(z_0)g'''(z_0)}{(g''(z_0))^2}$$

[You should check this against the special case where $f(z) \equiv 1$ and $g(z) = az^2 + bz^3$ for some $a, b \in \mathbb{C}^\times$.] $\diamond$

Exercise 7.3. Suppose that $p, q \in \mathbb{R}[x]$ are polynomials so that $\mathrm{DEG}(q) \geq \mathrm{DEG}(p) + 2$ and so that q has no roots in $\mathbb{R}$. Take $h = p/q$ to be the resulting rational function.

- Prove that $I = \int_{-\infty}^{\infty} h \, dx$ converges.

- Prove that p and q have unique holomorphic extensions, P and Q, to all of $\mathbb{C}$. Prove that P and Q are polynomials.

- Take $H = P/Q$. Prove that $I = 2\pi i \cdot \sum \mathrm{RES}(H, w)$; here the sum is over the zeros of Q in the upper half plane. $\diamond$

Exercise 7.4. Compute (via complex analysis) the following improper integral:

$$I_2 = \int_{-\infty}^{\infty} \frac{dz}{(1 + x^2)^2}$$

[Remember to check convergence "before begining", and to check your answer "after finishing".] $\diamond$

Exercise 7.5. Suppose that $a > 0$ is real. Compute (via complex analysis) the following improper integrals:

$$I_C(a) = \int_{-\infty}^{\infty} \frac{\cos(ax)}{1 + x^2} \, dx \qquad I_S(a) = \int_{-\infty}^{\infty} \frac{\sin(ax)}{1 + x^2} \, dx$$

[Remember to check convergence "before begining", and to check your answer "after finishing".] $\diamond$

Exercise 7.6. [Not examinable] Compute (via complex analysis) the following improper integrals, for $n \geq 1$:

$$I_n = \int_{-\infty}^{\infty} \frac{dz}{(1 + x^2)^n}$$ $\diamond$