Please let me (Saul) know if any of the problems are unclear or have typos.

Exercise 4.1. Suppose that U is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $P \subset U$ is a *polygon*: a region which is homeomorphic to a closed disk and having ∂P a union of line segments. Prove, using only euclidean geometry and Goursat's lemma, that $\int_{\partial P} f\,dz = 0$. [Of course this also follows from "Cauchy's theorem for regions".] ◇

Exercise 4.2. Suppose that U is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Suppose that $\gamma\colon [0,1] \to U$ is a contour. Suppose that $\epsilon > 0$ is a real number. Prove that there are real numbers

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = 1$$

so that, taking $p_k = \gamma(t_k)$, we have

a) for all k the line segment $[p_k, p_{k+1}]$ lies in U and

b) $\left|\int_\gamma f\,dz - \sum_k \int_{[p_k,p_{k+1}]} f\,dz\right| < \epsilon$.

[That is: there is a piecewise linear approximation to γ that, when integrated along, gives a good approximation to $\int_\gamma f\,dz$. Note that we do not assume that f is holomorphic.] ◇

Exercise 4.3. Suppose that U is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $B = B(z_0, R)$ is a disc with $\overline{B} \subset U$. Prove, using "only" the convexity of the circle and Exercises 4.1 and 4.2, that $\int_{\partial B} f\,dz = 0$. [There are two more "lightweight" proofs in the handwritten lecture notes. Of course this also follows from "Cauchy's theorem for regions".] ◇

Exercise 4.4. For each domain U and each holomorphic function f, either find a primitive F for f in U, or argue that no such primitive exists.

a) $U = \mathbb{C}$ and $f(z) = z^2$.

b) $U = \mathbb{C}$ and $f(z) = \textrm{Exp}(z)$.

c) $U = \mathbb{C}^\times$ and $f(z) = 1/z$.

d) $U = \mathbb{C}^\times$ and $f(z) = 1/\textrm{Exp}(z)$.

e) $U = \mathbb{C}^\times$ and $f(z) = \textrm{Exp}(1/z)$. ◇

Exercise 4.5. Define $\omega = \textrm{Exp}(2\pi i/6)$. Let $H \subset \mathbb{C}$ be the convex hull of the powers of ω. (So H is a regular hexagon inscribed in the unit circle.) Draw a picture of H. Equip H with a triangulation consisting of four positively oriented affine triangles. Draw these as well. Decorate the edges of each triangle with their induced orientations. Finally draw the induced orientation on ∂H. ◇

Exercise 4.6. Suppose that U is a domain. Suppose that $\sigma^2\colon \Delta^2 \to U$ is a singular two-simplex. Fix $\epsilon > 0$. Prove that there is some $N > 0$ with the following property. Suppose that $\textrm{Sub}^{(N)}(\sigma^2)$ is the N–fold midpoint subdivision of σ^2. Suppose that σ^2_k a singular two-simplex appearing as a summand of $\textrm{Sub}^{(N)}(\sigma^2)$. Then the diameter of the image of σ^2_k is less than ϵ. ◇

Exercise 4.7. Suppose that U is a domain. Prove that the composition of boundary operators

$$\partial_1 \circ \partial_2 \colon C_2(U) \to C_0(U)$$

is the zero homomorphism. Deduce that the group of one-boundaries $B_1(U)$ is contained in the group of one-cycles $Z_1(U)$. ◇

Exercise 4.8. Suppose that U is a convex domain in a real vector space V. Fix $n > 0$ a natural number. Fix z_n a point of U. Suppose that $\sigma = \sigma^{n-1} \colon \Delta^{n-1} \to U$ is a singular $(n-1)$-simplex. Define $\textsc{Con}(\sigma) \colon \Delta^n \to U$, the *cone over* σ *with apex* z_n, as follows:

$$\textsc{Con}(\sigma)(x_0, x_1, \ldots, x_n) = \begin{cases} z_n, & \text{if } x_n = 1 \\ x_n z_n + (1 - x_n)\sigma\left(\frac{x_0}{1-x_n}, \frac{x_1}{1-x_n}, \ldots, \frac{x_{n-1}}{1-x_n}\right), & \text{if } x_n \neq 1 \end{cases}$$

Prove that $\textsc{Con}(\sigma)$ is continuous. ◇