Please let me (Saul) know if any of the problems are unclear or have typos.

Exercise 3.1. For each (oriented) contour in Figure 3.2, label the complementary regions with their winding numbers. ◇

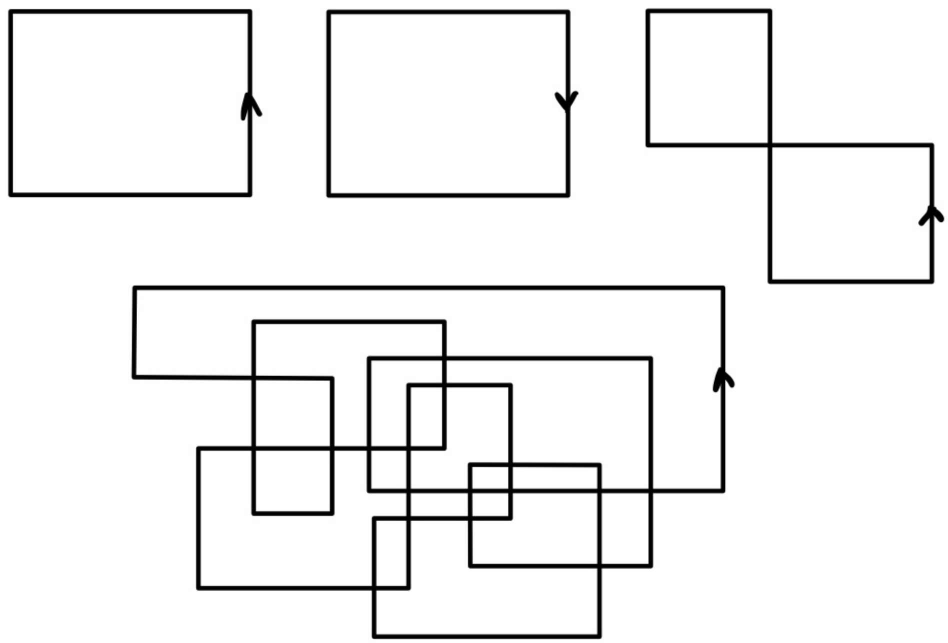

Figure 3.2: Four piecewise-linear contours.

Exercise 3.3. Suppose that $f(z) = 1/(1 - z + z^2)$. Find the series expansion of f about $z_0 = 0$ as well as its radius of convergence. ◇

Exercise 3.4. Suppose that $\mathrm{Exp}, \mathrm{Cos}, \mathrm{Sin} \colon \mathbb{C} \to \mathbb{C}$ are the complex exponential and the complex trigonometric functions, each given by its power series. Prove the following directly from the definitions.

a) $\mathrm{Exp}(0) = 1$.

b) $\mathrm{Exp}(z + w) = \mathrm{Exp}(z)\,\mathrm{Exp}(w)$.

c) $\mathrm{Exp}(iz) = \mathrm{Cos}(z) + i\,\mathrm{Sin}(z)$.

d) $\mathrm{Exp}(z) = \mathrm{Exp}(w)$ if and only if $z - w \in 2\pi i\mathbb{Z}$.

e) $\mathrm{Exp}(\mathbb{C}) = \mathbb{C}^\times$. ◇

Exercise 3.5. Take $\omega = \mathrm{Exp}(2\pi i/3)$; this is a primitive third root of unity. Let $T = T(1, \omega, \omega^2)$ be the triangle with vertices at 1, ω, and ω^2. Let $[1, \omega]$ be the line segment from 1 to ω, parametrised by $t \mapsto 1 - t + t\omega$. Define and parametrise $[\omega, \omega^2]$ and $[\omega^2, 1]$ in similar fashion. For each contour γ, compute the contour integral $\int_\gamma \frac{dz}{z}$ directly from the definitions.

a) $\gamma = [1, \omega]$ b) $\gamma = [\omega, \omega^2]$ c) $\gamma = [\omega^2, 1]$ d) $\gamma = \partial T$ $\diamond$

Exercise 3.6. Give a detailed proof that "holomorphic functions integrate to zero about rectangles". (Proofs were sketched in lecture on 2025-10-21 and in the lecture notes in Solution 6.7.) $\diamond$