Please let me (Saul) know if any of the problems are unclear or have typos.

Exercise 2.1. Take $\omega = e^{\pi i/4}$. Define $E\colon \mathbb{C} \to \mathbb{C}$ to be the rotation $E(z) = \omega z$. Define $D\colon \mathbb{C} \to \mathbb{C}$ by $D(z) = 2z/(1 - z^2)$. Define $\gamma\colon \mathbb{R} \to \mathbb{C}$ by $\gamma(t) = D(E(t))$. Sketch the image of γ. In your sketch, include the real and imaginary axes, indicate the orientation of γ, and label points of interest (including the images of the integers). [This exercise was inspired by Figure 8 of the article *Reflections on the lemniscate of Bernoulli: the forty-eight faces of a mathematical gem* by Langer and Singer.] ◇

Exercise 2.2. Suppose that $\gamma\colon [0, 2\pi] \to \mathbb{C}$ is the usual parametrisation of the unit circle: $\gamma(\theta) = e^{i\theta}$. For each function $f\colon \mathbb{C} \to \mathbb{C}$ below, sketch the contour $f \circ \gamma$. In your sketch, include the real and imaginary axes, indicate the orientation of the contour $f \circ \gamma$, and label points of interest (including the images of the multiples of $\pi/2$).

a) $f(z) = z$

b) $f(z) = 1/z$

c) $f(z) = 1/z^3$

d) $f(z) = z + 1/z$

e) $f(z) = 1 + z + z^2$

f) $f(z) = (z^2 + 1)/(z^2 + 2z - 1)$

g) $f(z) = (z^2 + 2z - 1)/(z^2 + 1)$

h) $f(z) = e^{Rz}$ for various real R in $[1, 100]$. What is happening near the origin? ◇

Exercise 2.3. Let $G = H \cup V$ be the following grid of segments in the unit square:

$$H = \{z \in \mathbb{C} \mid \textsc{Real}(z) \in [0, 1], \textsc{Imag}(z) \in \{0, 1/3, 2/3, 1\}\}$$
$$V = \{z \in \mathbb{C} \mid \textsc{Imag}(z) \in [0, 1], \textsc{Real}(z) \in \{0, 1/3, 2/3, 1\}\}$$

For each function $f\colon \mathbb{C} \to \mathbb{C}$ below, sketch $f(G)$. In your sketch, include the real and imaginary axes, draw $f(H)$ in blue, draw $f(V)$ in red, and label points of interest.

a) $f(z) = z^2$ b) $f(z) = z^3$ c) $f(z) = (iz)^3$

d) $f(z) = \sqrt{z}$ e) $f(z) = 1/z$ f) $f(z) = e^{2\pi i z}$ ◇

Exercise 2.4. Suppose that $f, g\colon \mathbb{C} \to \mathbb{C}$ are holomorphic. Prove, directly from the definitions, that $f + g$, $f \cdot g$, f/g, and $f \circ g$ are holomorphic. (For f/g we must avoid the zeros of g.) ◇

Exercise 2.5. Prove that $f(z) = \bar{z}$ is not holomorphic. Using this, or otherwise, prove that $\textsc{Real}(z)$, $\textsc{Imag}(z)$, and $|z|^2$ are not holomorphic. ◇