Please let me (Saul) know if any of the problems are unclear or have typos.

Exercise 1.1. We define the real-valued trigonometric functions sine and cosine by their power series.

$$\cos(x) = \sum_k (-1)^k \frac{x^{2k}}{(2k)!}$$
$$\sin(x) = \sum_k (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

From this, prove the following.

1. $\cos(0) = 1$ and $\sin(0) = 0$.
2. Cosine is even; sine is odd.
3. The derivative of cosine is negative sine; the derivative of sine is cosine.
4. The addition laws:
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y)$$
5. The circle law: $\cos^2(x) + \sin^2(x) = 1$.
6. Prove that cosine has a positive root. Denote this by x_0. Prove that
$$1.4 < \sqrt{2} < x_0 < \sqrt{6 - 2\sqrt{3}} < 1.6$$
[Challenge: In fact x_0 is *much* closer to the upper bound. Why? Further challenge: Find more decimal places, directly from the definitions.]
7. Prove that $\sin(x_0) = 1$.
8. Prove that
$$\cos(x + x_0) = -\sin(x) \qquad \text{and} \qquad \sin(x + x_0) = \cos(x)$$
9. Prove that
$$\cos(x + 4x_0) = \cos(x) \qquad \text{and} \qquad \sin(x + 4x_0) = \sin(x)$$
Thus $4x_0$ is a *period* of cosine (and of sine).
10. Prove that $4x_0$ is the smallest period of cosine (and of sine).

We now *define* $\pi = 2x_0$. Define $\gamma\colon [0, 2\pi] \to \mathbb{R}^2$ by $\gamma(\theta) = (\cos(\theta), \sin(\theta))$.

11. Prove that γ is a unit-speed parametrisation of the unit circle. Deduce that γ is injective. Deduce that the arc-length of the unit circle is 2π. $\diamond$

Exercise 1.2. Suppose that z and w are complex numbers. Prove $|z + w| \geq |z| - |w|$. $\diamond$

Exercise 1.3. Suppose that $C = C(0; 1)$ is the unit circle, oriented anticlockwise. Compute the contour integral $I(n) = \int_C (\bar{z})^n \, dz$ where n is an integer. $\diamond$

Exercise 1.4. We quote the entirety of [Thomas Clausen, Aufgabe 53, J. Reine Angew. Math. 2 (1827), 286–287].

> *Aufgabe. Wenn e die Basis der hyperbolischen Logarithmen, π den halben Kreisumfang, und n eine positive oder negative ganze Zahl bedeuten, so ist bekanntlich* $e^{2n\pi\sqrt{-1}} = 1$, $e^{1+2n\pi\sqrt{-1}} = e$*; folglich auch* $e^{(1+2n\pi\sqrt{-1})^2} = e = e^{1+4n\pi\sqrt{-1}-4n^2\pi^2}$*. Da aber* $e^{1+4n\pi\sqrt{-1}} = e$ *ist, so würde daraus folgen:* $e^{-4n^2\pi^2} = 1$*, welches absurd ist. Nachzuweisen, wo in der Herleitung dieses Resultats gefehlt ist.*

Here is a very loose translation. For any integer n we have $e^{2\pi i n} = 1$. Multiplying both sides by e we obtain $e^{1+2\pi i n} = e$. We now compute as follows.

$$\begin{aligned} e &= e^{1+2\pi i n} \\ &= \left(e^{1+2\pi i n}\right)^{1+2\pi i n} \\ &= e^{(1+2\pi i n)^2} \\ &= e^{1+4\pi i n - 4\pi^2 n^2} \\ &= e^{1-4\pi^2 n^2} \end{aligned}$$

Dividing both sides by e we deduce that 1 tends to zero as n tends to infinity. This is absurd. So, find (and explain) the mistake. $\diamond$