Lectures on Complex Analysis

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1 Introduction

Complex analysis is one of the classical branches in mathematics with roots in the 19th century and even earlier. Complex analysis, in particular the theory of conformal mappings, has many physical applications and is also used throughout analytic number theory. In modern times, it has become very popular through a new boost from complex dynamics and the pictures of fractals produced by iterating holomorphic functions. Another important application of complex analysis is in string theory which studies conformal invariants in quantum field theory.

1.1 A little history We begin by describing the evolution of the subject.

Figure 1: Khwarizmi, Cardano, Descartes, de Moivre, Euler

The study of complex numbers arose from attempts to find solutions to polynomial equations. Al-Khwarizmi (780-850) in his book Algebra (Al-Jabr) had solutions to quadratic equations of various types. [1] He was appointed as the astronomer and head of the House of Wisdom by Abbasid Caliph al-Ma'mun (reigned 813-833) in Baghdad, the contemporary capital city of the Abbasid Caliphate.

The first to solve the polynomial equation $x^3 + px = q$ was Scipione del Ferro (1465-1526). On his deathbed, del Ferro confided the formula to his pupil Antonio Maria Fiore, who subsequently challenged another mathematician Nicola "Tartaglia" Fontana (1500-1557) to a mathematical contest on solving cubics. [2] The night before the contest, Tartaglia rediscovered the formula and won the contest. Tartaglia in turn told the formula (but not the proof) to an influential mathematician Gerolamo Cardano (1501-1576), provided he signed an oath to secrecy. However, from a knowledge of the formula, Cardano was able to reconstruct the proof. Later, Cardano learned that del Ferro, not Tartaglia, had originally solved the problem and then, feeling under no further obligation towards Tartaglia, proceeded to publish the result in his Ars Magna (1545). [3] Cardano was also the first to introduce complex numbers $a + \sqrt{b}$ into algebra, but had misgivings about it. In the Ars magna he observed, for example, that the problem of finding two numbers

[1]Al-Jabr was translated into Latin by the English scholar Robert of Chester in 1145, was used until the 16th century as the principal mathematical textbook of many European universities.

[2]The name Tartaglia means "stammerer" a symptom of injuries acquired aged 12 during the french attack on his home town of Bresca

[3]Being multi-talented, he also invented the combination lock. Hoiwever, he was frequently short of money and kept himself solvent by being an accomplished gambler and chess player.

that add to 10 and multiply to 40 was satisfied by $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ but regarded the solution as both absurd and useless. [4]

Rene Descartes (1596-1650), the mathematician and philospher coined the term imaginary: "For any equation one can imagine as many roots [as its degree would suggest], but in many cases no quantity exists which corresponds to what one imagines." [5]

Abraham de Moivre (1667-1754), a protestant, left France to seek religious refuge in London at eighteen years of age. There he befriended Isaac Newton. [6] In 1698 he mentions that Newton knew, as early as 1676 of an equivalent expression to what is today known as de Moivre's theorem (and is probably one of the best known formulae) which states that:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

where n is an integer. [7]

Leonhard Euler (1707-1783) introduced the notation $i = \sqrt{-1}$ in his book *Introductio in analysin infinitorum* in 1748, and visualized complex numbers as points with rectangular coordinates, but did not give a satisfactory foundation for complex numbers. In contrast, there are indications that Carl Friedrich Gauss (1777-1855). had been in possession of the geometric representation of complex numbers since 1796, but it went unpublished until 1831, when he submitted his ideas to the Royal Society of Gottingen. It was Gauss who introduced the term complex number.

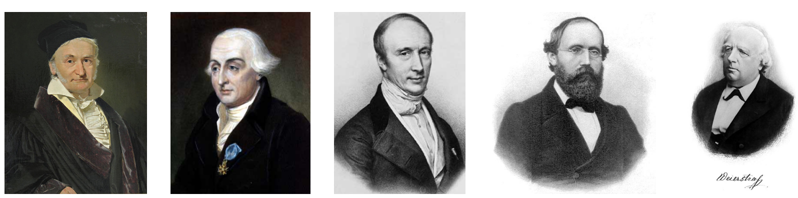


Figure 2: Gauss, Lagrange, Cauchy, Riemann, Weierstrass

Joseph-Louis Lagrange (1736-1813) showed that a function is analytic if it has a power-series expansion However, it is Augustin-Louis Cauchy (1789-1857) who really initiated the modern theory of complex functions in an 1814 memoir submitted to the French Academie des Sciences. Although the term analytic function was not mentioned in his memoir, the concept is present there. The memoir was eventually published in 1825. In particular, contour integrals appear in this memoir (although Poisson had written a 1820 paper with a path not on the real line). Cauchy also gave proofs of the Fundamental Theorem of Algebra (1799, 1815) which, as we will see, has an analytic proof . In summary, Cauchy, gave the foundation for most of the modern ideas in the field, including:

1. integration along paths and contours (1814);

2. calculus of residues (1826);

[4] Cardano was also said to have correctly predicted the exact date of his own death (but it has also been claimed that he achieved this by committing suicide!).

[5] Descartes was invited by Queen Christina of Sweden to Stockholm to give her lessons. However, after several meetings at 5am in her draughty castle he contracted pneumonia and died.

[6] According to a possibly apocryphal story, Newton, in the later years of his life, used to refer people posing mathematical questions to him to de Moivre, saying, "He knows all these things better than I do."

[7] De Moivre, like Cardan, is famed for predicting the day of his own death (27 November 1754). He found that he was sleeping 15 minutes longer each night and summing the arithmetic progression, calculated that he would die on the day that he slept for 24 hours.

3. integration formulae (1831);
4. Power series expansions (1831); and
5. applications to evaluation of definite integrals of real functions

The Cauchy- Riemann equations (actually dating back to d'Alembert 1752, then Euler 1757, d'Alembert 1761, Euler 1775, Lagrange 1781) are also usually attributed to Cauchy 1814-1831 and Riemann 1851.

Cauchy resigned from his academic positions in France in 1830 rather than to swear an oath of allegence to the new government. However, he felt able to resume his career in France in 1848, when the oath was finally abolshed.

Regarding subsequent work, Karl Weierstrauss (1815-1897) formulated analyticity in terms of existence of a complex derivative, which is the perspective taken in most textbooks, and Georg Riemann (1826-1866) made fundamental use of the notion of conformality (previously studied by Euler and Gauss). Later contributions were made by Poincaré to conformal maps and Teichmüller to quasi-conformal maps.

One of the most famous complex functions is the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

This function plays an important role in analytic number theory and the proof of the prime number theorem by Hadamard in 1896. Almost a century later there was an even shorter proof (using complex analysis) by Newman. In 1959 Riemann conjectured that the zeros of $\zeta(s)$ are only $\{-2n : n \in \mathbb{N}\}$ or have real part equal to $\frac{1}{2}$. This remains one of the major unsolved problems in mathematics.

1.2 A little notation We now recall some basic notation. We denote by $\mathbb{C}$ the complex numbers. If $z \in \mathbb{C}$ then we can write $z = x + iy$ where $x, y \in \mathbb{R}$. In particular, we represent $i = \sqrt{-1}$ and then $i^2 = -1$. [8]

We denote the real and imaginary parts by $x = Re(z)$ and $y = Im(z)$. We then write the absolute value as $|z| = \sqrt{x^2 + y^2}$. The following bounds are easily proved.

Exercise 1.1. *A few useful inequalities include: For $z_1.z_2 \in \mathbb{C}$:*

1. $|z_1 + z_2| \leq |z_1| + |z_2|$, *ie, the triangle inequality.*
2. $|z_1 z_2| \leq |z_1|.|z_2|$.
3. $||z_1| - |z_2|| \leq |z_1 - z_2|$.

Often it is convenient to use the identification of $\mathbb{C}$ with $\mathbb{R}^2$ to illustrate complex numbers. In particular, we can identify $z = x + iy$ with $\binom{x}{y}$. Occasionally it is useful to write complex numbers in radial coordinates, i.e., $z = re^{i\theta}$ where $r > 0$ and $0 \leq \theta < 2\pi$. In particular, if $z = x + iy$ then $r = \sqrt{x^2 + y^2}$ and (provided $z \neq 0$) $\theta = arg(z) = \tan^{-1}(y/x) \in (-\pi, \pi]$. Then $x = r\cos\theta$ and $y = \sin\theta$.

However, the function $arg(z)$ is not continuous on the negative real axis $\{z = x + iy : y \leq 0\}$. However, the function is continuous when restricted to the complement, i.e., the cut plane

$$\mathbb{C} - \{z = x + iy : y \leq 0\}.$$

We denote by $\overline{z} = x - iy = re^{-i\theta}$ the complex conjugate.

Exercise 1.2. *The following identities for $z = x + iy$ are easily checked.*

[8] This notation was used by Euler in 1777.

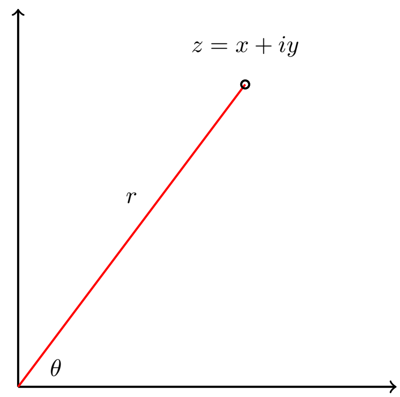


Figure 3: Radial coordinates (r, θ) correspond to the cartesian coordinates (x, y) for z.

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
2. $\overline{z_1 z_2} = \overline{z_1 z_2}$
3. $x = Re(z) = \frac{1}{2}(z + \bar{z})$
4. $y = Im(z) = \frac{1}{2i}(z - \bar{z})$
5. $|z|^2 = z\bar{z}$
6. $\bar{z} \in \mathbb{R}$ *iff* $z = \bar{z}$

If $z, w \in \mathbb{C}$ then we write

$$[z, w] = \{\alpha z + (1 - \alpha)w : 0 \leq \alpha \leq 1\}$$

for the line segment joining them.

Exercise 1.3. *Show that the map* $\phi : \mathbb{C} \to M_2(\mathbb{R})$ *(*$= 2 \times 2$ *real matrices)*

$$\phi : x + iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

is a monomorphism.

1.3 Roots of a complex number Given a complex number $w = x + iy = r(\cos\theta + i \sin\theta) = re^{i\theta}$ the nth roots $z \in \mathbb{C}$ satisfy $z^n = w$. They take the form

$$z_k = r^{1/n} \exp\left(i \frac{(\theta + 2\pi k)}{n}\right) \text{ for } k = 0, 1, \cdots, n - 1$$

1.4 The logarithm and powers The number $w = \xi + i\eta$ is a logarithm of $z = x + iy = re^{i\theta}$ if

$$e^{\xi + i\eta} = x + iy.$$

There are multiple solutions and the logarithm $\log z$ as the multiple element family

$$\{\log r + i(\theta + 2\pi n) : n \in \mathbb{Z}\}.$$

We can define the powers as follows. Let $z = x + iy = re^{i\theta}$. Then given $n \in \mathbb{Z}$ we can take the power

$$z^n = r^{1/n} e^{in\theta}$$

If $w = \xi + i\eta$ then we can define $z^w = e^{w \log z}$. In particular, we have that

$$z^w = \{e^{\xi \log r - \eta(\theta + 2\pi n) + i(\eta \log r + \xi(\theta + 2\pi n))} : n \in \mathbb{Z}\}.$$

Example 1.4. We can write $i^i = \{e^{-(\pi/2 + 2n\pi)} : n \in \mathbb{Z}\}$.

1.5 Some useful reference books

1. R. Churchill and J. Brown, Complex Variables and Applications (ISBN 0-07-010905-2). This is a fairly readable account including much of the material in the course.

2. I. Stewart and D. Tall, Complex Analysis (ISBN 0-52-128763-4). This is a popular and accessible book.

3. L. Alhfors, Complex Analysis: an Introduction to the Theory of Analytic Functions of One Complex Variable (ISBN 0-07-000657-1). This is a classic textbook, which contains much more material than included in the course and the treatment is fairly advanced.

4. S. Krantz and R. Greene, Function Theory of One Complex Variable (ISBN 0-82-183962-4). This is a nice textbook, which contains much more material than included in the course.

5. S. Krantz, Complex Analysis: The Geometric Viewpoint (0-88-385035-4). The first chapter gives a nice summary of some of the ideas in the course. The rest of the book is very interesting, but too geometric for this course.

2 The Riemann sphere and Mobius maps

2.1 The Riemann sphere It is convenient to add an extra point to $\mathbb{C}$ which we will denote by the symbol ∞. In order to accommodate this extra point ∞, we need to extend the complex plane by adding this point. One advantage is that we can then interpret $1/0 = \infty$ and $1/\infty = 0$. We denote by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere.

We will adopt the conventions that for $z \in \widehat{\mathbb{C}}$: $z/\infty = 0$; $z/0 = \infty$ if $z \neq 0$; $z + \infty = \infty + z$; and $z\infty = \infty z$ if $z \neq 0$.

Definition 2.1. There is a natural *stereographic projection*

$$\pi : \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x^2 + x_2^2 + x_3^2 = 1\} - \{(1, 0, 0)\} \to \mathbb{C}$$

between the unit sphere (minus the "north pole" $N = (1, 0, 0)$) and the complex plane defined by

$$\pi(x_1, x_2, x_3) = z := \left(\frac{x_1}{1 - x_3}\right) + i\left(\frac{x_2}{1 - x_3}\right).$$

9

In particular,

- 0 it the image of the "south pole" $(0, 0, -1)$ of the unit sphere.
- The unit circle $|z| = 1$ is the image of the equator $x_3 = 0$ of the Riemann sphere , i.e., $\pi(\{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\})$ coincides with the circle in the complex plane $\{z \in \mathbb{C} : |z| = 1\}$.
- The image $\pi(\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1, x_3 \leq 0\})$ of the lower hemisphere in the unit sphere is the unit disk in $\mathbb{C}$ and the image of the upper hemisphere minus the north pole $\pi(\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1, x_3 \geq 0\} - \{(0, 0, 1)\})$ is the complement of the unit disk.

[9] Ptolemy (AD 125) constructed such a map to plot the positions of heavenly bodies. Stereographic projection was known to Hipparchus of Nicaea, c.190BC - c.120BC.

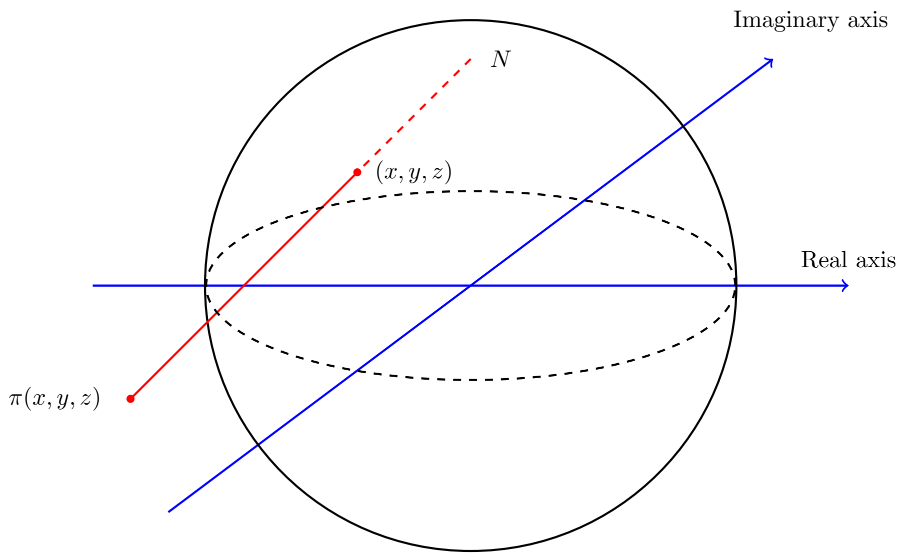


Figure 4: The stereographic projection π

We can characterise circles on the unit sphere as follows.

Definition 2.2. A *circle on the unit sphere* corresponds to the intersection of the sphere $x_1^2 + x_2^2 + x_3^2 = 1$ with a plane $ax_1 + bx_2 + cx_3 = d$.

The following relates such circles on the unit sphere to circles in the complex plane.

Theorem 2.3. *The stereographic projection of a circle on the sphere is either a circle or a line in* $\mathbb{C}$.

Proof. The image $z = \frac{x_1}{1-x_3} + i\frac{x_2}{1-x_3}$ of points (x_1, x_2, x_3) in the intersection under the projection satisfies

$$z + \overline{z} = \frac{2x_1}{1 - x_3}, z - \overline{z} = \frac{2ix_2}{1 - x_3} \text{ and } |z|^2 = \frac{(x_1^2 + x_2^2)}{(1 - x_3)^2} = \frac{1 - x_3^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3}.$$

and so

$$|z|^2 + 1 = \frac{2}{1 - x_3} \text{ and } |z|^2 - 1 = \frac{2x_3}{1 - x_3}$$

and we deduce that

$$(x_1, x_2, x_3) = \left(\frac{z + \overline{z}}{1 + |z^2|}, \frac{z - \overline{z}}{1 + |z^2|}, \frac{|z|^2 - 1}{1 + |z^2|}\right).$$

We therefore have that

$$a(z + \overline{z}) - ib(z - \overline{z}) + c(|z|^2 - 1) = d(|z|^2 + 1).$$

If we write $z = x + iy$ then

$$(d - c)(x^2 + y^2) - 2ax - 2by + (d + c) = 0. \tag{1}$$

Case I: If $c = d$ then the first term in (1) vanishes and this is the equation of a straight line in $\mathbb{C}$.

Case II: If $c \neq d$ then (1) can be first rewritten as

$$x^2 + y^2 - \frac{2ax}{d-c} - \frac{2by}{d-c} + \frac{(d+c)}{d-c} = 0$$

by completing the square, which we can further rearrange as

$$\left(x - \frac{a}{d-c}\right)^2 + \left(y - \frac{b}{d-c}\right)^2 = \frac{a^2 + b^2 + (c^2 - d^2)}{(d-c)^2}.$$

It only remains to show that $a^2 + b^2 + c^2 - d^2 > 0$ to see this is the equation of a circle centred at $z_0 = \frac{a}{d-c} + i\frac{b}{d-c} \in \mathbb{C}$. However, to see this we observe from the equation of the circle on the sphere

$$|d| = |ax_1 + bx_2 + cx_3| \leq \underbrace{\sqrt{x_1^2 + x_2^2 + x_3^2}}_{=1} \sqrt{a^2 + b^2 + c^2} \tag{2}$$

by the usual Cauchy-Schwartz inequality and since (x_1, x_2, x_3) lies on the unit circle. This completes the proof. □

Note that in the proof there is an equality in (2) only when $(a, b, c) = \lambda(x_1, x_2, x_3)$ for some λ, i.e., the plane is tangent to the sphere.

Exercise 2.4. *Prove the converse, i.e., the preimage under π of a circle or a straight line in $\mathbb{C}$ is a circle on the Riemann sphere.*

Proof of exercise 2.4. The equation of an arbitrary circle in the complex plane, centred at $z_0 = x_0 + iy_0$ and of radius $r > 0$ and consists of those $z = x + iy$ such that

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

We want to pick a, b, c, d so that the corresponding plane $(x_1, x_2, x_3).(a, b, c) = d$ induces this circle under stereographic projection corresponding to

$$\left(x - \frac{a}{d-c}\right)^2 + \left(y - \frac{b}{d-c}\right)^2 = \frac{a^2 + b^2 + c^2 - d^2}{(d-c)^2}$$

Therefore we can choose

$$a = x_0, b = y_0, c = \frac{1}{2}(x_0^2 + y_0^2 - r^2 - 1) \text{ and } d = \frac{1}{2}(x_0^2 + y_0^2 - r^2 + 1).$$

and the result follows. □

2.2 Metric on the Riemann sphere If we let $z, w \in \widehat{\mathbb{C}}$ then we can put a metric $d_{\widehat{\mathbb{C}}}(z, w) = \|\pi^{-1}(z) - \pi^{-1}(w)\|_2$ corresponding to the Euclidean norm $\|\cdot\|_2$ on the unit sphere in $\mathbb{R}^3$.

Exercise 2.5. *If $z := \pi((x_1, x_2, x_3))$ and $w := \pi((x_1', x_2', x_3'))$ then show that $\widehat{\mathbb{C}} \cup \{\infty\}$ is given by*

$$d_{\widehat{\mathbb{C}}}(z, w) := d_{\mathbb{R}^3}\left((x_1, x_2, x_3), (x_1', x_2', x_3')\right) = \frac{|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}$$

where $(x_1, x_2, x_3), (x_1', x_2', x_3') \neq (0, 0, 1)$ (i.e., $z, w \neq \infty$). Moreover, when $w = \infty$ we have

$$d_{\widehat{\mathbb{C}}}(z, \infty) := d((x_1, x_2, x_3), (0, 0, 1)) = \frac{2}{\sqrt{1 + |z|^2}}.$$

Proof of exercise 2.5. For $(x_1, x_2, x_3), (x_1', x_2', x_3') \neq (0,0,1)$ on the unit sphere:

$$\begin{aligned} d_{\mathbb{R}^3}\left((x_1,x_2,x_3),(x_1',x_2',x_3')\right) &= \sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2} \\ &= \sqrt{2(1-x_1x_1'+x_2x_2'+x_3x_3')}. \end{aligned}$$

Moreover, if $z := \pi((x_1,x_2,x_3))$ and $w := \pi((x_1',x_2',x_3'))$ we can write

$$x_1x_1'+x_2x_2'+x_3x_3' = .\frac{(z+\bar{z})(z'+\bar{z}')-(z-\bar{z})(z'-\bar{z}')+(|z|^2-1)(|z'|^2-1)}{(1+|z|^2)(1+|z'|^2)}$$

and thus we can rearrange these expressions to get

$$\sqrt{2(1-x_1x_1'+x_2x_2'+x_3x_3')} = \frac{|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}$$

where $(x_1,x_2,x_3),(x_1',x_2',x_3') \neq (0,0,1)$ (i.e., $z, w \neq \infty$). Moreover, when $w = \infty$ a similar argument gives

$$d_{\widehat{\mathbb{C}}}(z,\infty) := d((x_1,x_2,x_3),(0,0,1)) = \frac{2}{\sqrt{1+|z|^2}}.$$

Remark 2.6. To associate the appropriate topology to $\widehat{\mathbb{C}}$ we take the usual open sets in $\mathbb{C}$ plus the complements of compact sets union with the point ∞. This means that we can interpret $z_n \to z$ in the usual sense if $z \neq \infty$. However, we say that $z_n \to +\infty$ if for every $K > 0$ we have there exists $N > 0$ such that $|z| > K$. Then $\mathbb{C}$ is homeomorphic to the ball minus the "north pole".

In particular, the associated topology on $\widehat{\mathbb{C}}$ is such that a sequence $z_n \to +\infty$ as $n \to +\infty$ precisely when $1/z_n \to 0$.

2.3 Möbius maps. Mobius maps are a special class of bijections on the Riemann sphere $\widehat{\mathbb{C}}$. Sometimes they are sometimes called linear fractional transformations.

Figure 5: August Möbius (1790-1868)

We now define how to associate to four complex numbers a Möbius map.

Definition 2.7. Let $a, b, c, d \in \mathbb{C}$ with $ad \neq bc$ then we define an associated *Möbius map* $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$f(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \mathbb{C} - \{-\frac{d}{c}\} \\ \infty & \text{if } z = -\frac{d}{c} \\ \frac{a}{c} & \text{if } z = \infty \end{cases}$$

In particular, one can see that the function f is continuous with respect to the metric on the Riemann sphere $\widehat{\mathbb{C}}$.

Remark 2.8. We omit the degenerate special case $ad = bc$ since if we write

$$f(z) = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}$$

then if either $c = 0$ or $d = 0$ then $f(z) = \frac{a}{c}$ for all $z \in \widehat{\mathbb{C}}$. On the other hand, if $c = d = 0$ then $f(z)$ is simply not defined.

Remark 2.9. We can also assume without loss of generality that $ad - bc = 1$, since we see that replacing a, b, c, d by $\lambda a, \lambda b, \lambda c, \lambda d$ for any $\lambda \in \mathbb{C} - \{0\}$ gives the same map and we can take $\lambda = \frac{1}{\sqrt{ad-bc}}$. We will adopt this convention *when it is convenient.*

Lemma 2.10. *If $c \neq 0$ then the Möbius map f is invertible, and its inverse is also a Möbius map (and thus $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism).*

Proof. If $f(z) = \frac{az+b}{cz+d}$ we can assume without loss of generality that $ad - bc = 1$ by Remark 2.9. We can define the new Möbius map $f^{-1}(z) = \frac{dz-b}{-cz+a}$. We can then explicitly check $f \circ f^{-1}(z) = z$ by writing

$$\begin{aligned} f(f^{-1}(z)) &= \frac{a\left(\frac{dz-b}{-cz+a}\right)+b}{c\left(\frac{dz-b}{-cz+a}\right)+d} \\ &= \frac{a\,(dz-b)+b\,(-cz+a)}{c\,(dz-b)+d\,(-cz+a)} \\ &= \frac{(ad-bc)\,z}{(ad-bc)} = z. \end{aligned}$$

In particular, $f \circ f^{-1}$ is the identity Möbius map. Similarly, we can show $f^{-1} \circ f(z) = z$. □

Lemma 2.11. *If f_1 and f_2 are Möbius maps then so is the composition $f_1 \circ f_2$*

Proof. This follows immediately by substitution. If $f_1(z) = \frac{az+b}{cz+d}$ and $f_2(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ then

$$\begin{aligned} f_1 \circ f_2(z) = f_1(f_2(z)) &= \frac{af_2(z)+b}{cf_2(z)+d} \\ &= \frac{a\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)+b}{c\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)+d} = \frac{(a\alpha+b\gamma)z+a\beta+b\delta}{(c\alpha+d\gamma)z+c\beta+d\delta} \end{aligned}$$

□

In particular, the two previous lemmas show that the set of Möbius maps forms a group.

2.4 Properties of Möbius maps. We can write any Möbius map as a composition of simpler Möbius maps, which we now describe.

Definition 2.12. There are four fundamental examples of Möbius transformations $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

a) Translations: $z \mapsto z + b$ (i.e., $z \mapsto \frac{1.z+b}{0.z+1}$) where $b \in \mathbb{C}$. These are called *parabolic transformations.*

b) Rotations: $z \mapsto az$ (i.e, $z \mapsto \frac{\sqrt{a}.z+0}{0.z+\sqrt{a}^{-1}}$) where $|a| = 1$. These are called *elliptic transformations.*

c) Expansions (or Contractions): $z \mapsto \lambda z$ (*i.e.*,$z \mapsto \frac{\sqrt{\lambda}.z+0}{0.z+\sqrt{\lambda}^{-1}}$) with real $\lambda > 1$ (or $0 < \lambda < 1$). These are called *hyperbolic transformations.*

d) Inversions: $z \mapsto \frac{1}{z}$ (i.e., $z \mapsto \frac{0.z+0}{1.z+0}$).

We can use this to deduce the following.

Lemma 2.13. *Every Möbius map can be written as a composition of Möbius maps of the above type.*

Proof. Every Mobius map is a combination of three types of maps:

1. $f(z) = Az$ with $A \in \mathbb{C} - \{0\}$ (combining b) and c) above),
2. $f(z) = z + B$ (from a) above) where $B \in \mathbb{C}$, and
3. $f(z) = 1/z$ (from d) above)

Since we can write a given Möbius map as

$$\frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) + \left(b - \frac{ad}{c}\right)}{cz+d} = \frac{a}{c} + \frac{\left(b - \frac{ad}{c}\right)}{cz+d}$$

we see that this is a composition of these three types of map as follows:

$$z \mapsto cz \mapsto cz+d \mapsto \frac{1}{cz+d} \mapsto \left(b - \frac{ad}{c}\right)\left(\frac{1}{cz+d}\right) \mapsto \left(b - \frac{ad}{c}\right)\left(\frac{1}{cz+d}\right) + \frac{a}{c}$$

□

The following is an important property of Möbius maps. Consider circles in the complex plane of the form $C := \{z \in \mathbb{C} : |z - z_0| = r\}$ where $z_0 \in \mathbb{C}$ and $r > 0$.

Theorem 2.14 (Images of circles and straight lines)**.** *The image of a circle or straight line in $\mathbb{C}$ under a Möbius transformation is a circle or straight line.*

Proof. It suffices to consider three different types of Möbius transfomations:

1. If $f(z) = Az$ where $A \in \mathbb{C}$ with $A \neq 0$ then the image $f(C)$ is a circle.
2. If $f(z) = z + B$ where $B \in \mathbb{C}$ then the image $f(C)$ is a circle.
3. If $f(z) = 1/z$ then the image $f(C)$ is a circle.

The result is clear in cases (1) and (2). It remains to consider case (3). [10] We first observe $z = x + iy$ lies on a circle $C(z_0, r)$ centred at $z_0 = x_0 + iy_0$ of radius $r > 0$ precisely when

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

which we can expand as

$$x^2 + y^2 + \alpha x + \beta y + \gamma = 0. \tag{1}$$

where $\alpha = -2x_0$, $\beta = -2y_0$ and $\gamma = x_0^2 + y_0^2 - r^2$.

[10] The proof below is slightly different to the lectures. It has the advantage that it uses a more familiar formula for circles

If we set $w = \frac{1}{z}$ with $w = u + iv$ then

$$x = Re(1/w) = \frac{u}{u^2 + v^2} \text{ and } y = Im(1/w) = -\frac{v}{u^2 + v^2}. \tag{2}$$

Substituting (2) into (1) and simplifying gives

$$\begin{aligned} &\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{v}{u^2+v^2}\right)^2 + \alpha\left(\frac{u}{u^2+v^2}\right) + \beta\left(\frac{v}{u^2+v^2}\right) + \gamma \\ &= \frac{1}{u^2+v^2} + \frac{\alpha u}{u^2+v^2} + \frac{\beta v}{u^2+v^2} + \gamma = 0. \end{aligned} \tag{3}$$

If $\gamma \neq 0$ then multiplying (3) through by $(u^2 + v^2)/\gamma$ gives

$$1/\gamma + (\alpha/\gamma)u - (\beta/\gamma)v + u^2 + v^2 = 0$$

which is the equation of a(nother) circle. If $\gamma = 0$ then multiplying (3) through by $(u^2 + v^2)$ is the equation of a straight line (i.e., it is affine in u and v).

□

2.5 Existence and Uniqueness of Möbius maps The next result shows that Möbius maps are uniquely determined by a triple of distinct points.

Theorem 2.15. *For distinct points $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ and distinct points $w_1, w_2, w_3 \in \widehat{\mathbb{C}}$ there exists a unique Möbius map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the form*

$$f(z) = \frac{az + b}{cz + d} \text{ with } ad - bc = 1$$

such that $f(z_i) = w_i$, for $i = 1, 2, 3$.

Proof. We first prove existence and then uniqueness.

Existence. Consider the case that $z_1, z_2, z_3 \neq \infty$. Let

$$S(z) = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}$$

then we see that $S(z_1) = 1$, $S(z_2) = 0$, $S(z_3) = \infty$. Similarly, if we consider

$$T(z) = \frac{(z - w_2)(z_1 - w_3)}{(z - w_3)(z_1 - w_2)}$$

then we see that $T(w_1) = 1$, $T(w_2) = 0$, $T(w_3) = \infty$. Therefore, if we define $f(z) := S^{-1} \circ T(z)$, which is again a Möbius map, then we can then observe that $f(z_i) := T^{-1} \circ S(z_i) = w_i$ $(i = 1, 2, 3)$, as required.

In the case that one or two of the points are ∞ we merely need to change the Möbius maps used in the proof as follows:

1. In the case that $z_1 = \infty$ then we let $S(z) = \frac{z-z_2}{z-z_3}$.
2. In the case that $z_2 = \infty$ then we let $S(z) = \frac{z_1-z_3}{z-z_3}$.
3. In the case that $z_3 = \infty$ then we let $S(z) = \frac{z-z_2}{z_1-z_2}$.
4. In the case that $w_1 = \infty$ then we let $S(z) = \frac{z-w_2}{z-w_3}$.
5. In the case that $w_2 = \infty$ then we let $S(z) = \frac{w_1-w_3}{w-w_3}$.
6. In the case that $w_3 = \infty$ then we let $S(z) = \frac{w-w_2}{w_1-w_2}$.

This completes the proof of the existence of the Möbius map.

Uniqueness. We can assume without loss of generality that $w_1 = 1$, $w_2 = 0$, $w_3 = \infty$. (Otherwise we can additionally compose with a Möbius map $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ taking w_1, w_2, w_3 to $1, 0, \infty$, respectively, whose existence comes from the first part of the proof. We can apply the following argument to show that $f_1 \circ g^{-1} = f_2 \circ g^{-1}$ (which are Möbius maps by Lemma 2.11) which therefore gives us $f_1 = f_2$).

We can now assume that $f_j : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are two Möbius maps ($j = 1, 2$) such that $f_j(1) = z_1$, $f_j(0) = z_2$, $f_j(\infty) = z_3$. Since $S(z) = f_1^{-1} \circ f_2(z)$ is again a Möbius transformation (by Lemma 2.11) we can write

$$S(z) = \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{C}$. Moreover, by construction $f_1^{-1} \circ f_2 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixes the three points $1, 0, \infty$. In particular, we see that

$$\begin{aligned} S(0) &= b/d = 0 \implies b = 0 \\ S(\infty) &= a/c = \infty \implies c = 0, \text{ and} \\ S(1) &= (a + b)/(c + d) = a/d = 1 \end{aligned}$$

from which we deduce that $S(z) = z$ for all z, i.e., the identity map, and thus $f_1 = f_2$, as required. □

Let us consider an example.

Exercise 2.16. *Find the Möbius transformation f which maps $-1, 0, 1$ to the points $-i, 1, i$.*

Solution to Exercise 2.16. Assume that $f(z) = \frac{az+b}{cz+d}$. Since $f(0) = \frac{b}{d} = 1$ we have $b = d$ and so $f(z) = \frac{az+b}{cz+b}$. Similarly, since $f(-1) = \frac{-a+b}{-c+b} = -i \implies b - a = i(c - b)$ and $f(1) = \frac{a+b}{c+b} = i \implies ic + ib = a + b$. Adding the last two equations gives $2b = 2ic$ or equivalently $c = -ib$. Subtracting the two equations gives $-2a = -2ib$ or equivalently $a = ib$. Thus

$$f(z) = \frac{ibz + b}{-ibz + b} = \frac{iz + 1}{-iz + 1} = \frac{i - z}{i + z}.$$

(Formally, we might also multiply the coefficients by constant to get $ad - bc = 1$)

2.6 Fixed points

We would like to understand better the behaviour of different types of Möbius maps. An example is the following result.

Lemma 2.17. *Möbius maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ other than the identity map must fix at least one point and at most two.*

Proof. Let $f(z) = \frac{az+b}{cz+d}$ ($a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$). Assume that $f(z) = \frac{az+b}{cz+d} = z$ is a fixed point then this identity is equivalent to the quadratic equation $cz^2 + (d - a)z - b = 0$. The solution(s) to this quadratic equation are

$$z = \frac{(a - d) \pm \sqrt{(a - d)^2 + 4bc}}{2c} \in \mathbb{C}.$$

Unless f is the identity (i.e., $a = c$ and $b = d = 0$) this has two roots, with possibly a single root repeated if $(a - d)^2 + 4bc = 0$. □

3 Analytic functions

Let $f : U \to \mathbb{C}$ be a function defined on an open subset $U \subset \mathbb{C}$. We will give *three equivalent definitions* of analyticity. However, we first need to impose some additional conditions on U.[11]

3.1 Domains of functions. Assume that $U \subset \mathbb{C}$ is an open set, i.e., for every $z_0 \in U$ there exists $\epsilon > 0$ such that

$$B(z_0, \epsilon) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\} \subset U.$$

It is convenient to additionally assume that:

1. U is *(path) connected*, i.e., for any two points $z, w \in U$ we can find a continuous path $\gamma : [0, 1] \to U$ such that $\gamma(0) = z$ and $\gamma(1) = w$.
2. U is *simply connected*, i.e., any path $\gamma : [0, 1] \to U$ with $\gamma(0) = \gamma(1)$ can be contracted to a single point. (Equivalently, U is homeomorphic to the unit disk.)

This allows us to avoid remembering to assume these (when appropriate) in the hypotheses of results. [12]

Definition 3.1. A non-empty open subset $U \subset \mathbb{C}$ satisfying conditions 1 and 2 will be called a *domain*.

Henceforth we will always assume that U has these properties, unless we explicitly state otherwise.

3.2 Notation. We first need to introduce notation associated to each of the three equivalent definitions of analyticity we will present .

3.2.1 Notation for the first definition (power series)

Recall the following basic notation (perhaps more familiar for series with real coefficients).

Definition 3.2. Let $z_0 \in U$ and let $(a_n)_{n=0}^{\infty}$ be a sequence of complex numbers. We say that a series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ has a *radius of convergence* [13] $0 \leq R \leq +\infty$ at z_0 if

$$\frac{1}{R} = \limsup_{n \to +\infty} |a_n|^{1/n}.$$

Remark 3.3. The definition of radius of convergence is completely analogous to that for power series with real coffecients.

The following follows easily from the definition of the radius of convergence.

Lemma 3.4. *For any* $0 < r < R$ *(where* $R > 0$*) the series*

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

converges (uniformly) on $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

[11] Sometimes such functions are called holomorphic.

[12] In fact connectivity is a more a common assumption and the simply connected hypothesis is rarely used

[13] This is really a property of the sequence a_n and does not depend on z_0

Proof. We can choose N sufficiently large that $|a_n| \leq (\frac{r+R}{2})^n$ for $n \geq N$. Then for $z \in B(z_0, r)$ we have

$$|\sum_{n=0}^{\infty} a_n(z-z_0)^n| \leq \sum_{n=0}^{N-1} |a_n| r^n + \sum_{n=N}^{\infty} (\frac{2r}{r+R})^n < +\infty$$

□

We recall some familiar looking examples.

Example 3.5. We are used to Taylor series expansions of functions where $z \in \mathbb{R}$:

$$\begin{aligned} e^z &= 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \\ \sin(z) &= \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n-1)!} \text{ and} \\ \cos(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}. \end{aligned}$$

However these series correspond to power series for $z_0 = 0$ and converge for any $z \in \mathbb{C}$ (i.e., the radius of convergence for all of them is $r = \infty$).

The next result says that if the radius of convergence of a power series at some point is ∞ then the same is true for any other point

Lemma 3.6. *Let z_0, z_1 be two points in $\mathbb{C}$. If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has a radius of convergence $R = +\infty$ at z_0 then the same function is described by a series $\sum_{n=0}^{\infty} b_n(z-z_1)^n$ which has a radius of convergence $R = +\infty$ at z_1.*

Proof. Observe that for $z_0, z_1 \in \mathbb{C}$ then we can write

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(z-z_0)^n &= \sum_{n=0}^{\infty} a_n((z-z_1)+(z_1-z_0))^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(z-z_1)^k(z_1-z_0)^{n-k} \\ &= \sum_{m=0}^{\infty} b_m(z-z_1)^m \end{aligned}$$

where $b_m = \sum_{k=m}^{\infty} a_k \frac{(k+m)!}{k!m!}(z_1-z_0)^{k-m}$. But for any $\epsilon > 0$ there exists C such that $|a_k| \leq C\epsilon^k$ and thus $|b_m| \leq C \sum_{k=m}^{\infty} \epsilon^k (k+m)^m$ □

The following is useful in combining these definitions to get new functions.

Lemma 3.7. *If*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

both converge for any $z \in \mathbb{C}$ then the product function $(f.g)(z) = f(z)g(z)$ is given by a series $(f.g)(z) = \sum_{n=0}^{\infty} c_n z^n$ which converge for any $z \in \mathbb{C}$ and we can write

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

The proof is immediate.

3.2.2 Notation for the second definition (complex differentiability)

We begin by what it means for a function to be differentiable *as a complex function.*

Definition 3.8. Let U be a domain. We say that a function $f : U \to \mathbb{C}$ is *complex differentiable* at $z_0 \in U$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, i.e., $\forall \epsilon > 0\ \exists \delta > 0$ such that if $|z - z_0| < \delta$ then

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \le \epsilon |z - z_0|.$$

This condition is much more restrictive than the more familiar notion of differentiable functions on intervals in $\mathbb{R}$.

It is easy to see that one of the properties familiar in the real setting still holds for analytic functions:

Lemma 3.9. *f is complex differentiable at z_0 then it is continuous at z_0.*

Proof. Given $\epsilon > 0$ there exists $\delta > 0$ such that whenever $|z - z_0| < \delta$ then we have that

$$\left| f'(z_0) - \frac{f(z) - f(z_0)}{z - z_0} \right| < \epsilon.$$

In particular, for $|z - z_0| < \delta$ we can bound

$$|f(z) - f(z_0)| \le (|f'(z_0)| + \epsilon)|z - z_0| \le (|f'(z_0)| + \epsilon)\delta,$$

which can be made arbitrarily small by choosing $\delta > 0$ appropriately small. This is enough to give continuity. $\square$

3.2.3 Notation for the third definition (Cauchy-Riemann equations)

We write a function $f : U \to \mathbb{C}$ in the form $f(x + iy) = u(x, y) + iv(x, y)$ where $z = x + iy$ (with $x, y \in \mathbb{R}$) and $u(x, y), v(x, y) \in \mathbb{R}$.

Definition 3.10. We say that f satisfies the *Cauchy-Riemann equations* (at $z_0 = x_0 + iy_0 \in U$) if the partial derivatives

$$\frac{\partial u}{\partial x}(x_0, y_0), \frac{\partial v}{\partial y}(x_0, y_0), \frac{\partial u}{\partial y}(x_0, y_0) \text{ and } \frac{\partial v}{\partial x}((x_0, y_0)$$

all exist and satisfy the identities

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$
$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

We will also require in the sequel that these partial derivatives are continuous. [14]

Remark 3.11. Equivalently, recalling that $f(z) = u(x, y) + iv(x, y)$ we could write the Cauchy-Riemann equations concisely in the concise form

$$i\frac{\partial f}{\partial x}(z_0) = \frac{\partial f}{\partial y}(z_0)$$

where we recover the two identities in the definition above by considering the real and imaginary parts.

[14] One can give examples later which shows that this cannot be taken for granted.

3.3 Definitions of analyticity. We now give the following equivalent definitions of analyticity of a function.

Theorem 3.12. *The following three properties of a function $f : U \to \mathbb{R}$ are equivalent.*

1. *For all $z_0 \in U$ there exists $R > 0$ and $(a_n)_{n=0}^{\infty} \subset \mathbb{C}$ such that*
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
for $z \in B(z_0, r) \cap U$ and with radius of convergence R.
2. *$f(z)$ is complex differentiable at each $z_0 \in U$.*
3. *The partial derivatives*
$$\frac{\partial u}{\partial x}(x_0, y_0), \frac{\partial v}{\partial y}(x_0, y_0), \frac{\partial u}{\partial y}(x_0, y_0) \text{ and } \frac{\partial v}{\partial x}(x_0, y_0)$$
exist at each $z_0 = x_0 + iy_0 \in U$, **are continuous** *and satisfy the Cauchy-Riemann equations.*

Proof. **(1. $\Longrightarrow$ 2.)** Let $z_0 \in U$ and write
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
for $z \in B(z_0, r) \subset U$ for $0 < r < R$ sufficiently small, where $R > 0$ be the radius of convergence of the series.

Without loss of generality we can take $z_0 = 0$ (since this simplifies the notation without any loss of generality). We then claim that $f(z)$ is differentiable on $B(0, r)$ with derivative
$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$
To this end, given $z \in B(0, R)$ we can choose $|z| < \rho < r$ assume that for $h \in \mathbb{C}$ with $0 < |h| < r - |z|$ we have
$$\frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=1}^{\infty} a_n \underbrace{\left(\frac{(z+h)^n - z^n}{h} - n z^{n-1} \right)}_{=: g_n(h)} \tag{1}$$
where we denote
$$g_n(h) = \begin{cases} h \sum_{k=1}^{n-1} k (z+h)^{k-1} z^{n-k-1} & \text{if } n \neq 2 \\ 0 & \text{if } n = 1. \end{cases} \tag{2}$$
(noting that $(z+h)^n - z^n = ((z+h) - z)^2 \sum_{k=1}^{n} k z^{k-1} (z+h)^{n-k-1}$). In particular, we can bound
$$\sum_{k=1}^{n-1} k (z+h)^{k-1} z^{n-k-1} \leq \frac{n(n-1)}{2} \rho^{n-2}$$
provided h is sufficiently small that $|z + h| < \rho$. Thus
$$\begin{aligned} \left| \frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1} \right| &\leq h \sum_{n=2}^{\infty} |a_n| . |g_n(h)| \\ &\leq h \sum_{n=2}^{\infty} n^2 |a_n| \rho^{n-2} \end{aligned} \tag{4}$$

and since $\rho < R$ the last series in (4) converges. In particular, this implies that as $h \to 0$ the limit in the expression in (1) exists as h tends zero, i.e., $f(z)$ is complex differentiable at z_0. [15]

(2. $\implies$ 3.) We can write $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy \in U$ with $x, y \in \mathbb{R}$ and $u(x,y), v(x,y) : U \to \mathbb{R}$. Since $f(z)$ is complex differentiable at $z_0 \in U$ we have:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

where the limit is for small $h \in \mathbb{C}$. However, we can consider two particular special cases.

1. Assume that $h = t \in \mathbb{R}$ then

$$\lim_{t \to 0} \frac{f(z_0 + t) - f(z_0)}{t} = f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \qquad (a)$$

   where $z_0 = x_0 + iy_0$.

2. Assume that $h = it$ where $t \in \mathbb{R}$ then

$$\lim_{t \to 0} \frac{f(z_0 + it) - f(z_0)}{it} = f'(z_0) = -i\left(\frac{\partial u}{\partial y}(x_0, y_0) + i\frac{\partial v}{\partial y}(x_0, y_0)\right) \qquad (b)$$

   where $z_0 = x_0 + iy_0$.

Comparing the identities (a) and (b) we see from the real parts:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

and from the imaginary parts

$$\frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$$

We therefore conclude that the Cauchy-Riemann equations hold.

We still need that the derivatives are continuous (assuming that f is complex differentiable) which we will return to later once we have recalled Cauchy's theorem.

(3. $\implies$ 2.) We can (again) write $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$ and $u(x,y), v(x,y) : U \to \mathbb{R}$. where $z = x + iy \in U$.

Let $h = h_1 + ih_2$ (where $h_1, h_2 \in \mathbb{R}$) with $|h| < r$, where $r > 0$ is chosen sufficiently small that $B(z,r) \subset U$. We can then write

$$\begin{aligned} &u(x + h_1, y + h_2) - u(x,y) \\ &= (u(x + h_1, y + h_2) - u(x, y + h_2)) + (u(x, y + h_2) - u(x,y)) \end{aligned} \qquad (1)$$

(by adding in, and subtracting again, the term $u(x, y + h_2)$). By the usual Mean Value Theorem applied to the real valued functions $t \mapsto \frac{\partial u}{\partial x}(x + t, y + h_2)h_1$ and $s \mapsto \frac{\partial u}{\partial y}(x, y+s)h_2$ we can choose h_3, h_4 such that $0 \le |h_3| \le |h_1|$ and $0 \le |h_4| \le |h_2|$ such that

$$\begin{aligned} u(x + h_1, y + h_2) - u(x, y + h_2) &= \frac{\partial u}{\partial x}(x + h_3, y + h_2)h_1 \\ u(x, y + h_2) - u(x,y) &= \frac{\partial u}{\partial y}(x, y + h_4)h_2 \end{aligned} \qquad (2)$$

[15] More formally, by the Weierstass M-test that says that $\sum_n g_n(h)$ converges uniformly if $\|f\|_\infty \le M_n$, say, for which $\sum_n M_n < +\infty$.

For notational convenience we can now introduce

$$\Delta(h_1,h_2,h_3,h_4) := \\ h_1\left(\frac{\partial u}{\partial x}(x+h_3,y+h_2)-\frac{\partial u}{\partial x}(x,y)\right)+h_2\left(\frac{\partial u}{\partial y}(x,y+h_4)-\frac{\partial u}{\partial y}(x,y)\right). \tag{3}$$

Since $|h_1|,|h_2|\leq|h|$ (and thus $|h_3|,|h_4|\leq|h|$) and by continuity of the partial derivatives [16]

$$\lim_{h\to 0}\frac{1}{|h|}\left(\sup_{|h_1|,|h_2|,|h_3|,|h_4|\leq|h|}|\Delta(h_1,h_2,h_3,h_4)|\right)=0 \tag{4}$$

In particular, with $\Delta(h_1,h_2,h_3,h_4)$ we can now combine (1), (2) and (3) to write

$$u(x+h_1,y+h_2)-u(x,y)=h_1\frac{\partial u}{\partial x}(x,y)+h_2\frac{\partial u}{\partial y}(x,y)+\Delta(h_1,h_2,h_3,h_4).$$

(where, of course, h_3 and h_4 depend on h_1 and h_2). Similarly,

$$v(x+h_1,y+h_2)-v(x,y)=h_1\frac{\partial v}{\partial x}(x,y)+h_2\frac{\partial v}{\partial y}(x,y)+\Pi(h_1,h_2,h_3,h_4).$$

where $\Pi(h_1,h_2,h_3,h_4)$ is the analogue of $\Delta(h_1,h_2,h_3,h_4)$ and satisfies

$$\lim_{h\to 0}\frac{1}{|h|}\left(\sup_{|h_1|,|h_2|,|h_3|,|h_4|\leq|h|}|\Delta(h_1,h_2,h_3,h_4)|\right)=0 \tag{5}$$

By the above we can write

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{u(x+h_1,y+h_2)-u(x,y)}{h}+i\left(\frac{v(x+h_1,y+h_2)-v(z)}{h}\right)\\ &= \frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y)+\frac{\Delta(h_1,h_2,h_3,h_4)+i\Pi(h_1,h_2,h_3,h_4)}{h}\end{aligned}$$

Therefore by (4) and (5) and using the Cauchy-Riemann equations we deduce that $f(z)$ is differentiable with derivative[17]

$$f'(z)=\frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y)\left(=\frac{\partial v}{\partial y}(x,y)-i\frac{\partial u}{\partial y}(x,y)\right).$$

It also remains to show (2) $\implies$ (1) (which we will return to later once we have recalled Cauchy's theorem) □

Continuity of the partial derivatives in part 3 of this theorem is important as the following example shows.

Example 3.13. If $f:\mathbb{C}\to\mathbb{C}$ is defined by

$$f(x+iy)=\begin{cases}0 & \text{if } xy=0\\ 1 & \text{otherwise}\end{cases}$$

then all the partial derivatives appearing in the Cauchy-Riemann equations are zero, and thus the Cauchy-Riemann equations are trivially zero, but f is not differentiable (or even continuous, by Lemma 3.9).

[16]Note that this is where we use continuity of the partial derivatives

[17]The proof gives us the bonus that it identifies the complex derivative $f'(z)$

Example 3.14 (The exponential function)**.** We now have three ways to see that the exponential function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^z$ is analytic for every $z_0 = x_0 + iy_0 \in \mathbb{C}$.

1. We can write the exponential function as a power series

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!}$$

which has radius of convergence $R = \infty$.

2. The exponential function has a complex derivative at z_0 which takes the form $f'(z_0) = e^{z_0}$, i.e.,

$$\lim_{h \to 0} \frac{e^{z_0+h} - e^{z_0}}{h} = e^{z_0} \lim_{h \to 0} \frac{e^h - 1}{h} = e^{z_0}.$$

3. If $z = x + iy$ then we can write $f(z) = u(x, y) + iv(x, y)$ where $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$. Then

$$\begin{aligned} \frac{\partial u}{\partial x}(x_0, y_0) &= e^{x_0} \cos(y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) &= e^{x_0} \sin(y_0) = -\frac{\partial v}{\partial x}(x_0, y_0) \end{aligned}$$

i.e., the Cauchy-Riemann equations hold and the partial derivatives are continuous.

Exercise 3.15. *Show that for an analytic function $f : U \to \mathbb{C}$ any of the following implies that f is constant:*

1. *$Re(f)$ is constant;*

2. *$Im(f)$ is constant;*

3. *$|f|$ is constant;*

4. *$Arg(f)$ is constant.*

Proof. This is illustrated by the proof for 2. Suppose that $Im(f) = 0$, i.e., $f : U \to \mathbb{R}$ is real valued and analytic, then we want to see that f is constant. Since $f(z) = u(z) + iv(z) = u(z)$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \implies \frac{\partial u}{\partial x} = -i\frac{\partial u}{\partial y} = 0$$

(since one is real and the other imaginary). Integrating with y fixed we have that $u(x, y) = g(y)$ and $\frac{\partial g}{\partial y} = 0$. Thus $g(y) = C$ and hence $f(z) = C$.

□

3.4 The principle of isolated zeros and the identity theorem. We begin with a particularly useful property of analytic functions which makes use of the definition using power series.

Theorem 3.16 (The principle of isolated zeros)**.** *Let $f : U \to \mathbb{C}$ be a non-constant analytic function. The zeros $\mathcal{Z} = \{z : f(z) = 0\}$ cannot accumulate inside U, i.e., there doesn't exist $z_\infty \in Z$ and $z_n \in Z$ with $z_n \neq z_\infty$ $(n \geq 1)$ such that $z_n \to z_\infty$ as $n \to +\infty$.*

Proof. Assume for a contradiction there exists $z_\infty \in Z$ and $z_k \in Z$ with $z_k \neq z_\infty$ $(k \geq 1)$ such that $z_k \to z_\infty$ as $k \to +\infty$. We will use the following.

Claim. Writing $f(z) = \sum_{n=0}^{\infty} a_n (z - z_\infty)^n$ in a neighbourhood of z_∞ we have that $a_n = 0$ for $n \geq 1$. □

Proof of claim. Assume for a contraction there exists $N \geq 1$ such that $a_N \neq 0$ and $a_1 = a_2 = \cdots a_{N-1} = 0$. Since each $z_k \in Z$ we have

$$f(z_k) = 0 = \sum_{n=N}^{\infty} a_n (z_k - z_\infty)^n \tag{1}$$

If $R > 0$ is the radius of convergence and we choose $0 < r < R$ and $C > 0$ such that $|a_n| \leq Cr^n$ for $n \geq 1$. Thus for k sufficiently large $|z_\infty - z_k| < 1/r$ we have by (1) that

$$\begin{aligned} 0 = \left| \sum_{n=N}^{\infty} a_n (z_k - z_\infty)^n \right| &\geq |a_N| - C \sum_{n=N+1}^{\infty} (r|z_k - z_\infty|)^n \\ &\geq |a_N| - C \frac{(r|z_k - z_\infty|)^{N+1}}{1 - r|z_k - z_\infty|} \end{aligned} \tag{2}$$

Moreover, by choosing k even larger (if necessary) we have that the last term in (2) is strictly positive. This leads to a contradiction in (2) which proves the claim.

The claim tells us that $f(z) = a_0$, i.e., f is a constant function contradicting the assumption that f is non-constant on $B(z_\infty, R)$. □

This leads to the following.

Corollary 3.17 (Identity theorem)**.** *Let $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be analytic functions and assume that $E = \{z \in U : f(z) = g(z)\}$ contains an accumulation point then $f(z) = g(z)$, for all $z \in U$.*

Proof. The function $h : U \to \mathbb{C}$ defined by $h(z) = f(z) - g(z)$ is again analytic and vanishes on the set E. If $z_\infty \in E$ is not isolated then by the previous theorem h must on some ball $B(z_\infty, r)$, for some $r > 0$.

Let t $w \in U - B(z_\infty, r)$ and choose a continuous path $\gamma : [0, 1] \to U$ with $\gamma(0) = z_\infty$ and $\gamma(1) = w$. Let $0 < T \leq 1$ such that

$$T = \sup\{t \in [0, 1] : h(\gamma(s)) = 0 \text{ for } 0 \leq s \leq t\}.$$

We would like to conclude that $T = 1$ and thus $h(w) = h(\gamma(1)) = 0$. Assume for a contradiction that $T < 1$. By continuity of h and γ the point $w_\infty := \gamma(T) \in E$. Moreover, we can assume without loss of generality that w_∞ is an accumulation point of points in E (for example, $w_n = \gamma(T - 1/n)$, say, as $n \to +\infty$). However, we can then apply the previous theorem to deduce that h vanishes in a neighbourhood of w_∞. But this leads to a contradiction (to the definition of T) since by continuity of γ we could then choose $S > T$ with $h(\gamma(t)) = 0$ for $S > t > T$. □

3.5 Vanishing derivatives of analytic functions. The following result is analogous to the result for real valued functions on an interval that a vanishing derivative implies for function is constant. The proof makes use of the definition of analyticity in terms of the Cauchy-Riemann equations. [18]

[18] Here is it the connectedness of the domain which is used in a crucial way: Which must be the case since if an open set has disconnected components then clearly the result may be false, with functions taking (different) constant values on different components

Theorem 3.18. *If $f : U \to \mathbb{C}$ is analytic and $f'(z) = 0$ then f is a constant function.*

Proof. If $z \in U$ then by hypothesis and the formula 3.3 in the proof of the equivalence of the three definitions of analyticity we can write

$$f'(z) = 0 = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial x}.$$

Comparing real and imaginary parts gives that

$$0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

at every point $z \in U$.

Fix a point $z_0 = x_0 + iy_0 \in U$ and for $\epsilon > 0$ sufficiently small consider the open ball $B(z_0, \epsilon) \subset U$ contained in the domain U with centre $z_0$and radius $\epsilon > 0$

Given any other point $z_1 = x_1 + iy_1 \in B(z_0, \epsilon)$ then we can associate a third point $z_2 = x_1 + iy_0$ (i.e., $Re(z_1) = Re(z_0)$ and $Im(z_1) = Im(z_2)$). Consider the function $u : B(z_0, \epsilon) \to \mathbb{C}$ then we observe that

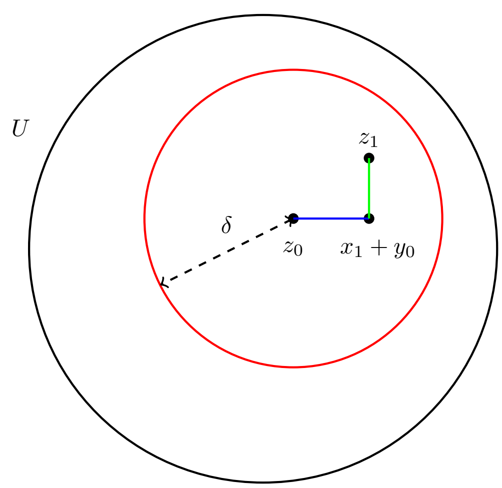
$$\frac{\partial u}{\partial x}(x, y_0) = 0, \forall x + iy_0 \in B(z_0, \epsilon) \implies u(x_0, y_0) = u(x_1, y_0) \tag{a}$$

since the vanishing partial derivative implies that the function $u(\cdot)$ is constant on horizonal lines (i.e., constant as x changes when y_0 stays constant) by integration with respect to x.

Similarly, we observe that

$$\frac{\partial u}{\partial y}(x_1, y) = 0, \forall x_1 + iy \in B(z_0, \epsilon) \implies v(x_1, y_0) = v(x_1, y_1) \tag{b}$$

since the vanishing partial derivative implies that the function $v(\cdot)$ is constant on vertical lines (i.e., constant as y changes providing x_1 stays constant) by integration with respect to y.



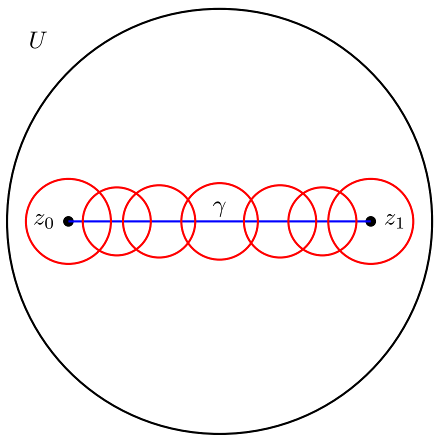
Thus combining (a) and (b) gives that $u(z_0, y_0) = u(x_1, x_1)$ and since z_1 was chosen arbitrarily we can deduce that u is constant on $B(z_0)$

Repeating the argument with v replacing u we can deduce that it too is constant on $B(z_0, \epsilon)$. Thus u, v, and therefore and $f = u + iv$, are all constant functions on any ball $B(z_0, \epsilon)$.

It remains to show that f is constant on all of U and this is where connectivity comes into play. Fix a point $z_0 \in U$ in the domain and choose any other point $z_1 \in U$. Let $\gamma : [0,1] \to U$ be a continuous path with $\gamma(0) = z_0$ and $\gamma(1) = z_1$. Let $0 < T \leq 1$ be such that

$$T = \sup\{t : f(\gamma(s)) = f(z_0) \text{ for } 0 \leq s \leq t\}$$

We would like to conclude that $T = 1$ and thus $f(z_0) = f(z_1)$. Assume for a contradiction that $T < 1$. By continuity of h and γ at the point $z' := \gamma(T)$ we have that $f(z') = f(z_0)$.



However, we can apply the argument with z' in place of z_0 to deduce that $f(z)$ is constant on some ball $B(z', \epsilon')$, with $\epsilon' > 0$ sufficiently small. But this leads to a contradiction (to the definition of T) since by continuity of γ and f we could then choose $S > T$ with $h(\gamma(t)) = 0$ for $S > t \geq 0$. □

3.6 Removable singularities. It is worth mentioning the following result which is sometimes useful. It basically says that if a function is analytic on a ball or domain except possibly at a "hole" in the middle (like a polo mint) then in fact it is analytic on the whole ball.

Theorem 3.19 (Removable Singularities Theorem)**.** *Assume $f : U \to \mathbb{C}$ is analytic expect possibly at $z_0 \in U$. However, if $f(z)$ is bounded on the ball $B(z_0, \epsilon) = \{z : |z - z_0| < \epsilon\} \subset U$, for some $\epsilon > 0$, then $f(z)$ is analytic at $z = z_0$ too.*

Proof. We can introduce a new function $g : U \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} (z - z_0)^2 f(z) & \text{if } \in B(z_0, \epsilon) - \{z_0\} \\ 0 & \text{if } z = z_0 \end{cases}$$

The function $g(z)$ is clearly analytic on $B(z_0, \epsilon) - \{z_0\}$ because of the way it is defined. Moreover, at the point z_0 we have

$$\frac{g(z) - g(z_0)}{z - z_0} = (z - z_0) f(z) \to 0 \text{ as } z \to z_0$$

(since if $|f(z)|$ is bounded on $B(z_0, \epsilon)$ by $M > 0$, say, then $|(z - z_0) f(z)| \leq M|z - a| \to 0$ as $z \to z_0$) showing that the complex derivative exists at z_0 with $g'(z_0) = 0$ and $g(z_0) = 0$. In particular, this means that $g(z)$ is complex analytic at z_0 too.

Using the definition of analyticity of $g(z)$ at z_0 in terms of power series [19] we can write

$$g(z) = \sum_{n=2}^{\infty} a_n (z - z_0)^n$$

[19]Notice that thus far in the proof we are using two of the equivalent definitions of analyticity

(where, $a_0 = g(z_0) = 0$ and $a_1 = g'(z_0) = 0$ and $\limsup_{n\to+\infty} |a_n|^{1/n} < +\infty$) on a small ball about z_0.

Let us set $f(z_0) = a_2$. Then from the power series expansion for $g(z)$ we can write

$$f(z) = g(z)/(z - z_0)^2 = \sum_{n=0}^{\infty} a_{n+2}(z - z_0)^n$$

which gives us a power series expansion for $f(z)$ at $z = z_0$. From this we deduce that f is analytic at $z = z_0$, as required. □

3.7 Harmonic functions Harmonic functions are real valued functions (and thus cannot be analytic) but which are nontheless closely related to analytic functions. We now use the definition of analyticity in terms of the Cauchy Riemann equations to relate analytic function to harmonic functions. We begin with a basic definition for real valued functions.

Definition 3.20. We say that a smooth function $h : U \to \mathbb{R}$ is harmonic if $\Delta h = 0$ where we denote

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

i.e., Δ is called a second order differential operator. This equation $\Delta h = 0$ is often called *Laplace's equation*.

Remark 3.21. The property of being harmonic is easily seen to be preserved by scaler multiples and linear combinations of harmonic functions are harmonic.

The connection to analytic functions is that if $f : U \to \mathbb{C}$ is an analytic function and we write $f(x+iy) = u(x, y)+iv(x, y)$ then since it satisfies the Cauchy-Riemann equations then differentiating

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

with respect to x implies that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} \tag{1}$$

and differentiating the identity

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

with respect to y implies that

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}. \tag{2}$$

In particular, adding (1) and (2) we see that u is harmonic (i.e., $\Delta u = 0$). A similar argument applied to v shows that it too is harmonic.

Definition 3.22. If $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}$ are both harmonic and satisfy the Cauchy-Riemann equations in u then v is called the *harmonic conjugate* of u.

Example 3.23. The function $f(z) = z^2$ is analytic and thus writing $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$ gives

$$f(z) = (u(x, y) + iv(x, y))^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

where we conclude $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. By the above discussion u and v are both harmonic. Furthermore, $2xy$ is the harmonic conjugate of $x^2 - y^2$.

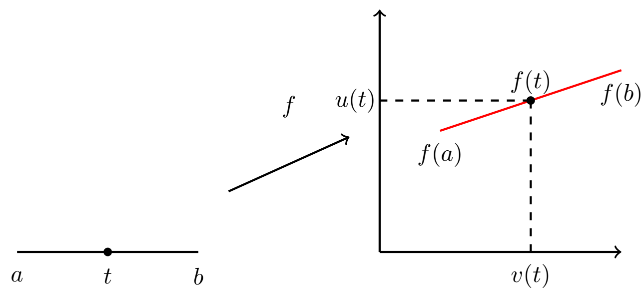
4 Contour Integrals and Cauchy's Theorem

The most important theorem in complex analysis we have already seen before. We bein with some standard defintions.

4.1 Contour integrals Given a continuous curve $\gamma : [a, b] \to \mathbb{C}$ we can take the real and imaginary parts to write $\gamma(t) = u(t) + iv(t)$, where $u, v : [a, b] \to \mathbb{R}$ are real valued.

Definition 4.1. If $f : [a, b] \to \mathbb{C}$ is continuous then we define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt.$$



For continuous functions $f, g : [a, b] \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$ we have from the definition that

$$\int_a^b (\alpha f(t) + \beta f(t))dt = \alpha \int_a^b f(t)dt + \beta \int_a^b f(t))dt$$

We also observe the following:

Lemma 4.2. $|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$

Proof. Assume that $\int_a^b f(t)dt \neq 0$ (otherwise the result is trivial) then we can write $re^{i\theta} = \int_a^b f(t)dt$ with $r > 0$ and $0 \leq \theta < 2\pi$. Thus

$$r = |\int_a^b f(t)dt| = \int_a^b \left(e^{-i\theta} f(t)\right) dt \in \mathbb{R}. \tag{1}$$

Therefore, we can write

$$\begin{aligned} r = Re \left(\int_a^b e^{-i\theta} f(t)dt \right) &= \int_a^b Re \left(e^{-i\theta} f(t)\right) dt \\ &\leq \int_a^b \left|e^{-i\theta} f(t)\right| dt = \int_a^b |f(t)|\, dt \end{aligned} \tag{2}$$

and thus comparing (1) and (2) we have the result $|\int_a^b f(t)dt| = r \leq \int_a^b |f(t)|\, dt$ □

We next make a simple but key observation.

Simple observation If $\phi : [c, d] \to [a, b]$ is differentiable map with continuous positive derivative $\phi'(t)$ for $(a \leq t \leq b)$, so that $\phi(t)$ is strictly increasing, then

$$\int_a^b f(s)ds = \int_c^d f(\phi(t))\phi'(t)dt$$

by the rule law and the change of variables with $s = \phi(t)$.

Definition 4.3. A *contour* (or *curve* or *arc* or *path*) γ is the image of a closed interval $[a, b]$ in $\mathbb{C}$ under a continuous map $z : [a, b] \to \mathbb{C}$ with an orientation or order provided by the increasing on $[a, b]$.

In particular $z(a)$ is the first point on the contour and $z(b)$ is the last point on the contour.

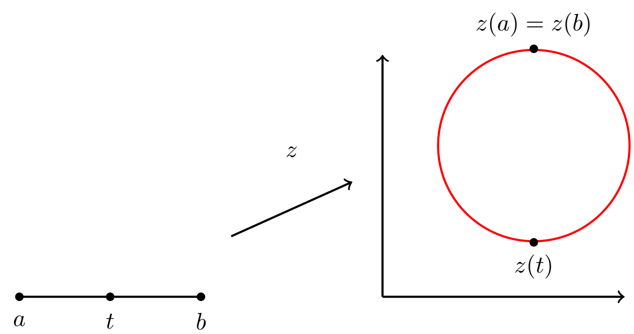
Definition 4.4. Any continuous map z giving the curve γ and its orientation is called a *parameterization.* The contour is called *smooth* if there is a parameterization function z which is

1. differentiable in its interval $[a, b]$ of definition and
2. $z'(t)$ is continuous on $[a, b]$.

The contour γ is called *piecewise smooth* if there exists a parameterization $z : [a, b] \to \mathbb{C}$ for the contour and a partition $a = a_0 < a_1 < \cdots < a_n = b$ such that $z|[a_i, a_{i+1}]$ is smooth. [20]

Finally, we want to impose some additional natural conditions on the contours.

Definition 4.5. A contour γ is called *simple* if it is given by an injective parameterization (i.e., $z(a) = z(b) \implies a = b$). A contour γ is *closed* if it is given by a parameterization $z : [a, b] \to$ (i.e., $z(a) = z(b) \implies a = b$ or $a = b$)



We will mainly be concerned with piecewise smooth curves and piecewise smooth parameterizations. Of course, any (piecewise) smooth curve γ may be given by many parameterisations, i.e., if $\phi : [c, d] \to [a, b]$ is strictly increasing then $w : [c, d] \to \mathbb{C}$ defined by $w(t) = z(\phi(t))$ is also piecewise smooth if z is piecewise smooth and ϕ is differentiable with continuous and positive derivative.

We say that two parameterizations z and w are equivalent if there exists such a ϕ.

Exercise 4.6. *Show that this is an equivalence relation on parameterizations*

Proof. Clearly we can write $w \circ \phi^{-1} = z$ and ϕ^{-1} is continuous with positive derivative thus the relation is reflexive. Transitivity follows simply since if $z_2(t) = z_1(\phi_1(t))$ and $z_3(t) = z_2(\phi_2(t))$ then $z_3(t) = z_1((\phi_1 \circ \phi_2)(t))$. □

Definition 4.7. We can also define the *reverse* of a parameterization. If $z : [a, b] \to \gamma \subset \mathbb{C}$ is a piecewise smooth parameterization of γ then we obtain a piecewise smooth parameterization of γ (in the reverse sense) by $z_- : [-b, -a] \to \mathbb{C}$ with $z_-(t) = z(-t)$. Let us denote by $-\gamma$ the reverse of γ (i.e., the same image curve but with the opposite parameterization).

[20] A piecewise smooth curve γ may be given by many parameterizations. If $\phi : [c, d] \to [a, b]$ is strictly increasing then $w; [c, d] \to \mathbb{C}$ defined by $w(t) = z(\phi(t))$

There is a natural way to combine suitable contours.

Definition 4.8. If γ_1 is piecewise smooth and γ_2 is piecewise smooth and if the last point of γ_1 is the same as the first point of γ_2 then we define $\gamma_1 \cup \gamma_2$ as follows. If $z_1 : [a, b] \to \gamma_1 \subset \mathbb{C}$ is a parameterization of γ_1 and $z_2 : [c, d] \to \gamma_2 \subset \mathbb{C}$ is a parameterization of γ_2 then we define a parameterization $z_3 : [a, b + d - c] \to \gamma_3 := \gamma_1 \cup \gamma_2$ by

$$z_3(t) = \begin{cases} z_1(t) & \text{if } t \in [a, b] \\ z_2(\phi(t)) & \text{if } t \in [b, b + d - c] \end{cases}$$

where ϕ is the translation $\phi(t) = t + (b - c)$.

If γ is a smooth contour parameterized by $z : [a, b] \to \mathbb{C}$ then we define

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Remark 4.9. If γ is a simple smooth curve parameterized by the smooth function $z : [a, b] \to \mathbb{C}$ then we can write $l(\gamma) = \int_a^b |z'(t)|dt$.

Lemma 4.10. *If γ is piecewise smooth and f is continuous on γ and is bounded by $|f(z)| \le M$ on γ then*

$$\left| \int_\gamma f(z)dz \right| \le Ml(\gamma).$$

Proof. Let $z : [a, b] \to \gamma \subset \mathbb{C}$ be a parameterization. Then

$$\begin{aligned} \left| \int_\gamma f(z)dz \right| &= \left| \int_a^b f(z(t))z'(t)dt \right| \\ &\le \int_a^b |f(z(t))|.|z'(t)|dt \le M \int_a^b |z'(t)|dt = Ml(\gamma). \end{aligned}$$

□

In general, we are interested in integration around a piecewise smooth simple contour. Given smooth curves $z : [a_i, a_{i+1}] \to \mathbb{C}$ for $a = a_0 < a_1 < \cdots < a_n = b$ and writing $\gamma_i = z_i([a_i, a_{i+1}])$ for $i = 0, 1, \cdots, n - 1$ we define

$$\int_\gamma f(z)dz = \sum_{i=0}^{n-1} \int_{\gamma_i} f(z)dz$$

once we define the integral for smooth contours we can then extend it to smooth contours.

Exercise 4.11. *Show that $\int_{-\gamma} f(w)dw = -\int_\gamma f(z)dz$*

Proof. Let $z : [a, b] \to \gamma \subset \mathbb{C}$ be a parameterization for the curve γ. Then $w : [-b, -a] \to -\gamma \subset \mathbb{C}$ given by $w(t) = z(-t)$ is a parameterization for the curve $-\gamma$. We can then write $\int_{-\gamma} f(w)dw = \int_{-b}^{-a} f(z(-t))z'(-t)dt$. Substituting $s = -t$ this becomes: $-\int_a^b f(z(s))z'(s)ds = -\int_\gamma f(z)dz$. □

Convention: In the case of simple closed curves the convention is that the orientation of the curve is anti-clockwise (or counter clockwise) unless otherwise stated.

Example 4.12. Let $\gamma = [w_1, w_2] = \{w_1(1-t) + w_2 t : 0 \leq t \leq 1\}$ and consider the smooth parameterization $z : [0,1] \to \mathbb{C}$ given by $z(t) = w_1(1-t) + w_2 t$. Then $z'(t) = (w_2 - w_1)$ and

$$\int_{[w_1,w_2]} f(z)dz = \int_0^1 f(w_1(1-t) + w_2 t)(w_2 - w_1)dt.$$

Consider the more specific cases that $f(z) = z^n$ with $n \in \mathbb{Z}$ with $n \neq -1$ then

$$\int_0^1 (w_1(1-t) + w_2 t)^n (w_2 - w_1)dt = \frac{w_2^{n+1}}{n+1} - \frac{w_1^{n+1}}{n+1}.$$

Example 4.13. Let γ be the θ-arc in the unit circle with parameterization $z : [0,\theta] \to \mathbb{C}$ given by $z(t) = e^{it}$. Then $z'(t) = ie^{it}$. Let $f(z) = z^n$ for $n \in \mathbb{Z}$.

1. If $n \neq -1$ then

$$\int_\gamma f(z)dz = \int_0^\theta e^{nit} i e^{int} dt = \frac{i}{(n+1)i}\left[e^{i(n+1)t}\right]_0^\theta = \frac{e^{i(n+1)\theta} - 1}{n+1}.$$

This is equal to zero if $\theta = 2\pi$.

2. If $n = -1$ then $\int_\gamma \frac{1}{z}dz = \theta i$. The particular case with $\theta = 2\pi$ gives us the important example of Cauchy's theorem

$$\int_\gamma f(z)dz = 2\pi i.$$

Remark 4.14. For our purposes it suffices to consider piecewise smooth curves. However, it is possible to define the integrals with respect to curves with less regularity. [21]

4.2 Goursat's Theorem. A useful result with an interesting proof is the following result due to Goursat.

Figure 6: Édouard Goursart (1858-1936)

[21] The integration with respect to arc length along a simple contour γ is rectifiable if it is given by a continuous map $z : [a,b] \to \mathbb{C}$ which is of bounded variation, i.e., the collection of numbers

$$\sum_{i=0}^{n-1} |z(a_{i+1}) - z(a_i)|,$$

ranging over all partitions $a = a_0 < a_1 < \cdots < a_n = b$, is bounded. The supremum of these numbers is by defintion the length of $l(\gamma)$. Now suppose that γ is smooth represented by $z : [a,b] \to \mathbb{C}$ then γ is rectifiable and $l(\gamma) = \int_a^b |z'(t)|dt$.

Theorem 4.15 (Goursat's Theorem)**.** *Let $f : U \to \mathbb{C}$ be analytic. Let $\gamma : [0,1] \to U$ be a piecewise smooth closed simple curve then $\int_\gamma f dt = 0$.*

Proof. We begin with a proof of this result in the case that γ is the boundary of a rectangle $R_0 = [a,b] \times [c,d]$, say. In particular

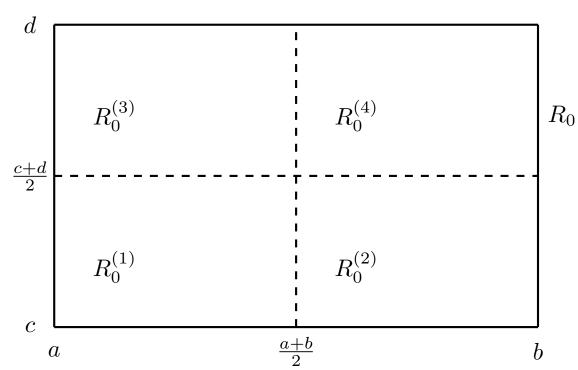
$$\gamma = \partial R_0 = \{a\} \times [c,d] \cup [a,b] \times \{d\} \cup \{b\} \times [c,d] \cup [a,b] \times \{c\}.$$

We can then subdivide R into four similar rectangles

$$\begin{aligned} R_0^{(1)} &= [a, a+b/2] \times [c, (c+d)/2] \\ R_0^{(2)} &= [(a+b)/2, b] \times [c, (c+d/2)] \\ R_0^{(3)} &= [a, a+b/2] \times [(c+d)/2, d] \\ R_0^{(4)} &= [(a+b)/2, b] \times [(c+d/2), d] \end{aligned}$$

and then we can write

$$\int_{\partial R_0} f(z)dz = \sum_{i=1}^{4} \int_{\partial R_0^{(i)}} f(z)dz.$$



By taking absolute values and using the triangle inequality we can write

$$\left| \int_{\partial R_0} f(z)dz \right| \leq 4 \max_{1 \leq i \leq 4} \left| \int_{\partial R_0^{(i)}} f(z)dz \right|. \tag{1}$$

Let R_1 be the subrectangle which corresponds to the largest of the four terms on the righthandside of (1). We can repeat this step iteratively to construct a sequence of subrectangles

$$R_0 \supset R_1 \supset R_s \supset \cdots$$

each of which is half the size of the previous rectangle and such that

$$\left| \int_{\partial R_0} f(z)dz \right| \leq 4^n \left| \int_{\partial R_n} f(z)dz \right|. \tag{a}$$

We can choose (the unique) value $z_0 \in \cap_{n=0}^{\infty} R_n$.

Since we are assuming $f : U \to \mathbb{C}$ is analytic we have that $f(z)$ is complex differentiable (at z_0). In particular, this means that for any $\epsilon > 0$ we an choose $\delta > 0$ so that providing $|z - z_0| < \delta$ we have that

$$|f(z) - f(z_0) + f'(z_0)(z - z_0)| \leq \epsilon |z - z_0|.$$

Provided that n is sufficiently large that the diameter

$$\text{diam}(R_n) = \sqrt{(b-a)+(d-c)}2^{-n}$$

of R_n is smaller than δ then we have that for all $z \in R_n$ we can write that

$$\begin{aligned} |f(z) - f(z_0) - f'(z_0)(z - z_0)| &\le \epsilon \text{diam}(R_n) \\ &= \epsilon\sqrt{(b-a)+(d-c)}2^{-n}. \end{aligned} \tag{b}$$

Thus since we see that that the length of the boundary ∂R_n has length

$$\text{length}(\partial R_n) = \sqrt{(b-a)+(d-c)}2^{-n}$$

and we can bound the integral

$$\begin{aligned} \int_{\partial R_n} (f(z) - f(z_0) - f'(z_0)(z - z_0))dz &\le \epsilon \text{diam}(\partial R_n)\text{length}(\partial R_n) \\ &\le \epsilon\sqrt{(b-a)+(d-c)}2^{-n}2^{-n}2((b-a)+(d-c)) \\ &= 2^{-2n+1}\epsilon\left((b-a)+(d-c)\right)^{3/2} \end{aligned}$$

Moreover, we have the following:

Lemma 4.16. *For each rectangle R_n ($n \ge 1$):*

1. *We can evaluate $\int_{\partial R_n} 1.dz = 0$ (where 1 is the constant function taking value 1);*
2. *We can evaluate $\int_{\partial R_n} (z - z_0)dz = 0$.*

Proof. The first part is easy, since the contribution to the integrals from opposite sides cancel. More precisely, if the rectange is $R_n = [a_n, b_n] \times [c_n, d_n]$ then we can use the paramerterizations

$$\begin{aligned} &z_1(t) = t + ic_n \text{ for } a_n \le t \le b_n \\ &z_2(t) = b_n + it \text{ for } c_n \le t \le d_n \\ &z_3(t) = (b_n + a_n) - t + ic \text{ for } a_n \le t \le b_n \\ &z_4(t) = a_n + i(c_n + d_n - t) \text{ for } c_n \le t \le d_n \end{aligned}$$

for the four line segments in the boundary ∂R_n of R_n. Thus we can compute

$$\begin{aligned} \int_{\partial R_n} 1.dz &= \int_{a_n}^{b_n} z_1'(t)dt + \int_{c_n}^{d_n} z_2'(t)dt + \int_{a_n}^{b_n} z_3'(t)dt + \int_{c_n}^{d_n} z_4'(t)dt \\ &= 1 + i - 1 - i = 0 \end{aligned}$$

For the second part, we begin by writing

$$\int_{\partial R_n} (z - z_0)dz = \int_{\partial R_n} zdz - \int_{\partial R_n} 1.dz = \int_{\partial R_n} zdz$$

since $\int_{\partial R_n} 1.dz = 0$ by the first part. Using the same parameterizations as above we can now write

$$
\begin{aligned}
\int_{\partial R_n} z dz &= \int_{a_n}^{b_n} z_1(t)z_1'(t)dt + \int_{c_n}^{d_n} z_2(t)z_2'(t)dt + \int_{a_n}^{b_n} z_3(t)z_3'(t)dt + \int_{c_n}^{d_n} z_4(t)z_4'(t)dt \\
&= 1 \times \int_{a_n}^{b_n} (t + ic_n)dt + i \times \int_{c_n}^{d_n} (b_n + it)dt \\
&- 1 \times \int_{a_n}^{b_n} (b_n + a_n - t + ic_n)dt - i \times \int_{c_n}^{d_n} (a_n + i(c_n + d_n - t)dt \\
&= \left(\frac{b_n^2}{2} - \frac{a_n^2}{2} + ic_n(b_n - a_n)\right) + i\left(i\frac{d_n^2}{2} - i\frac{c_n^2}{2} + b_n(d_n - c_n)\right) \\
&- \left(-\frac{b_n^2}{2} + \frac{a_n^2}{2} + (b_n + a_n - ic_n)(b_n - a_n)\right) - i\left(-\frac{d_n^2}{2} + \frac{c_n^2}{2} + (a_n + i(c_n + d_n))(d_n - c_n)\right) \\
&= 0
\end{aligned}
$$

□

We can now return to the proof of Goursat's theorem. Comparing (a) and (b) and the above lemma we can write:

$$
\begin{aligned}
\left|\int_{\partial R_0} f(z)dz\right| &\leq 4^n \left| f(z_0) \underbrace{\left(\int_{\partial R_n} 1 dz\right)}_{=0 \text{ by Lemma part 1}} + f'(z_0) \underbrace{\left(\int_{\partial R_n} (z - z_0)dz\right)}_{=0 \text{ by Lemma part 2}} \right| \\
&\quad + \epsilon_n \underbrace{\left(\sup_{z \in R_n} |z - z_0|\right) \operatorname{diam}(R_n)}_{\leq C.2^{-n}2^n} \\
&\leq 0 + C.\epsilon
\end{aligned}
$$

where $C = \left((b-a)^2 + (d-c)^2\right)^{3/2}$ Thus, since ϵ can be chosen arbitrarily small, we deduce that $\int_{\partial R_0} f(z)dz = 0$, as required. □

Remark 4.17. Given a simple closed curve γ_c given by the contacternation of horizontal and vertical lines in $\mathbb{C}$ we can write the integral

$$
\int_{\gamma_c} f(z)dz = \sum_{i=1}^{N} \int_{\gamma_i} f(z)dz = 0
$$

where γ_i are the boundaries of rectangles as above.

One interesting point about the proof of Gorusat's Theorem is that it only assumes that f is complex differentiable and, in particular, doesn't assume that the derivative is continuous.

Although the condition on the boundary curve seems very restrictive it leads to a general result with more general closed curves γ. We briefly outline the idea.

Assume for simplicity that $f : B(0,r) \to \mathbb{C}$ is analytic. it suffices to show that f has a primitive, i.e., an analytic function $F : B(0,r) \to \mathbb{C}$ such that $F'(z) = f(z)$. This is because for any smooth curve $\gamma_0 : [0,1] \to B(0,r)$ with $\gamma_0(0) = z_0$ and $\gamma_0(1) = z_1$ one can calculate

$$
\int_{\gamma} F'(\xi)d\xi = F(z_1) - F(z_0).
$$

To construct the primitive we can choose $z = x + iy \in B(0,r)$ and for $h > 0$ sufficiently small $z + h = (x+h) + iy \in B(0,r)$ and then let

$$F(z) := \int_0^x f(\xi)d\xi + \int_x^z f(\xi)d\xi.$$

Then

$$\begin{aligned} F(z+h) - F(z) &= \left(\int_0^{x+h} f(\xi)d\xi + \int_{x+h}^{z+h} f(\xi)d\xi \right) - \left(\int_0^x f(\xi)d\xi + \int_x^z f(\xi)d\xi \right) \\ &= \int_x^{x+h} f(\xi)d\xi + \int_{x+h}^{z+h} f(\xi)d\xi + \int_x^z f(\xi)d\xi \end{aligned} \tag{1}$$

However, by Goursat's theorem applied to the rectangle with corners x, $x+h$, z and $z+h$ we have

$$\int_x^{x+h} f(\xi)d\xi + \int_{x+h}^{z+h} f(\xi)d\xi + \int_{z+h}^{z} f(\xi)d\xi + \int_z^x f(\xi)d\xi = 0 \tag{2}$$

and substituting (2) into (1) gives that

$$F(z+h) - F(z) = \int_z^{z+h} f(\xi)d\xi.$$

This can then be used to show that

$$\frac{\partial F}{\partial x}(z) = \lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z). \tag{1}$$

By rewriting $F(z)$ using vertical paths followed by horizontal paths (using Goursat's theorem again) an analogous argument then shows that

$$\frac{\partial F}{\partial y}(z) = if(z). \tag{2}$$

Comparing (1) and (2) we see that F satisfies the Cauchy-Riemann equations and thus is analytic. Moreover, $F'(z) = f(z)$ as required.

The famous Cauchy Integral Theorem (or at least one version of it) is now a corollary of Goursat's theorem.

Theorem 4.18 (Cauchy Integral Theorem)**.** *Let $f : U \to \mathbb{C}$ be an analytic function. Assume that z_0 lies inside a piecewise smooth simple closed curve $\gamma \subset U$. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz \tag{1}$$

Before embarking on the proof we consider a special case. Let $f(z) = 1$ be identically equal to the constant value 1 and let $\gamma = C(z_0, \epsilon)$ for $\epsilon > 0$ sufficiently small. We have already seem by explicit computation that

$$\int_{C(z_0,\epsilon} \frac{f(z)}{z - z_0} dz = 2\pi i$$

Proof. We only need to apply the previous result of Goursat to a suitable analytic function. Consider $g : U \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \\ f'(z_0) & if z = 0 \end{cases}$$

This function is clearly analytic for $z \neq z_0$. At $z = z_0$ we see that $\lim_{z\to z_0} g(z) = f'(z_0)$ since we know that f is analytic (and this complex differentiable at z_0). In particular, $g : U \to \mathbb{C}$ is continuous on U (and thus by the removable singularity theorem it is also analytic on U). We can now apply Goursat's theorem to $g : U \to \mathbb{C}$ and γ to deduce that

$$\int_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz = \int_\gamma g(z)dz = 0$$

We can now rearrange this formula to write

$$\int_\gamma \frac{f(z)}{z - z_0} dz = \int_\gamma \frac{f(z_0)}{z - z_0} dz = f(z_0) \int_\gamma \frac{1}{z - z_0} dz = 2\pi i f(z_0)$$

where we use that

$$\int_\gamma \frac{1}{z - z_0} dz = 2\pi i$$

This was established before when $\gamma = C(z_0, \epsilon)$ and extends to these more general curves again using Goursat's theorem. □

Example 4.19 (Gauss Mean Theorem)**.** Let $f : U \to \mathbb{C}$ be analytic. Let $z_0 \in U$ and $\delta > 0$ such that $\overline{B(z_0, \delta} \subset U$. Then using the parameterization $z : [0, 2\pi) \to \mathbb{C}$ by $z(t) = z_0 + \delta e^{2\pi i t}$ we see that $z'(t) = 2\pi i \delta e^{2\pi i t}$ and therefore

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_{C(z_0,\delta))} \frac{f(z)}{z_0 - z} z'(t)dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \delta e^{2\pi i t})}{\delta e^{2\pi i t}} (2\pi i \delta e^{2\pi i t})dt = \int_0^{2\pi} f\left(z_0 + \delta e^{2\pi i t}\right) dt \end{aligned}$$

The following corollary shows that integrals between points are independent of the path.

Corollary 4.20. *Let $f : U \to U$ be analytic and U is connected and simply connected. Let $z_1, z_2 \in U$. Then for any piecewise smooth curve γ from z_1 to z_2 the integral $\int_\gamma f(z)dz$ is independent of γ.*

Proof. Let γ_1, γ_2 are two piecewise smooth curves leading from z_1 to z_2 contained in U then $\gamma_1 \cup -\gamma_2$ is a closed curve. Let us assume for simplicity that this is a simple closed curve and then by Goursat's theorem we can write

$$\int_{\gamma_1 \cup -\gamma_2} f(z)dz = 0 = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz$$

giving the required identity. If there are self-intersections in $\gamma_1 \cup -\gamma_2$ then we can write it as a union of simple closed curves (and possibly degenerating to arcs) and the argument applied to each of these to give the result. □

The following is a converse to Cauchy's theorem.

Corollary 4.21 (Moreira's Theorem)**.** *Let $f : U \to \mathbb{R}$ be continuous and assume that $\int_\gamma f(z)dz = 0$ for any simple closed curve γ then $f : U \to \mathbb{R}$ is analytic.*

Proof. Fix $z_0 \in U$ then we define a primitive function $F : U \to \mathbb{C}$ by $F(z) := \int_\gamma f(z)dz$ where $\gamma : [0, 1] \to \mathbb{C}$ is given by $\gamma(0) = z_0$ and $\gamma(1) = z$.

We can assume that $F(z)$ is independent of γ by the previous corollary since U is a domain which is connected and simply connected. Then we see that $F(z)$ is complex differentiable (and $F'(z) = f(z)$). We can then deduce that $F''(z) = f'(z)$ exists and is analytic. Thus f is complex differentiable at each $z \in U$ and therefore analytic. □

As a further corollary there is a formula for the (higher) derivatives. Let $C(a, r)$ be a circle of radius r centred at a (with a clockwise orientation)

Corollary 4.22. *Let $f : U \to \mathbb{C}$ be analytic. For z_0 inside γ we can write*

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^2} dz$$

and, more generally,for $k \geq 1$ the kth derivative takes the form

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{k+1}} dz$$

Proof. We can begin by differentating the identity in the Cauchy Integral Theorem to get

$$\begin{aligned} f'(z_0) = \frac{d}{d\xi} f(\xi)|_{\xi=z_0} &= \frac{d}{d\xi}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-\xi}dz\right)|_{\xi=z_0} \\ &= \frac{1}{2\pi i}\int_\gamma \frac{d}{d\xi}\left(\frac{f(z)}{z-\xi}dz\right)|_{\xi=z_0} \\ &= \frac{1}{2\pi i}\int_\gamma \left(\frac{f(z)}{(z-\xi)^2}dz\right)|_{\xi=z_0} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^2}dz \end{aligned}$$

which establishes the result for $n = 1$. Repeating this argument and differentiating more times gives the required result. ☐

4.3 Returning to the equivalence of the definitions of analyticity We still need to complete two parts of the proof of the equivalence of the definitions of analytic functions.

4.3.1 Writing complex differentiable functions as power series.

Let $f : U \to \mathbb{C}$ be a function which is complex differentiable at every point in the domain. Let $z_0 \in U$ and choose $\epsilon > 0$ such that $B(z_0, \epsilon) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\} \subset U$.

Choose any $z \in B(z_0, \epsilon)$ and then let $\rho := |z_0 - z| < \epsilon$. Choose $0 < r < \epsilon$ then by Cauchy's integral theorem

$$f(z) = \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{f(\xi)}{\xi - z} d\xi \tag{1}$$

where $C(z_0, r) = \{z \in \mathbb{C} : |z_0 - z| = r\}$. For any $\xi \in C(z_0, r)$ we can expand

$$\begin{aligned} \frac{1}{\xi - z} &= \frac{1}{\xi - z_0} \frac{1}{\left(1 - \left(\frac{z-z_0}{\xi-z_0}\right)\right)} \\ &= \frac{1}{\xi - z_0}\left(1 + \left(\frac{z-z_0}{\xi-z_0}\right) + \left(\frac{z-z_0}{\xi-z_0}\right)^2 + \cdots\right) \end{aligned} \tag{2}$$

where $\left|\frac{z-z_0}{\xi-z_0}\right| = \frac{\rho}{r} < 1$. Thus by (1) and (2) we can write

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + a_n(z - z_0)^n + \cdots$$

where

$$a_0 = \frac{1}{2\pi i}\int_{C(z_0,r)} \frac{f(\xi)}{\xi - z_0} d\xi,$$
$$a_1 = \frac{1}{2\pi i}\int_{C(z_0,r)} \frac{f(\xi)}{(\xi - z_0)^2} d\xi,$$
$$\vdots$$
$$a_n = \frac{1}{2\pi i}\int_{C(z_0,r)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi, \text{etc}$$

Thus $f(z)$ has a power series expansion which has a non-zero radius of convergence

$$R = \limsup_{n\to+\infty} |a_n|^{1/n} > r > 0$$

as required.

4.4 Continuity of partial derivatives. Let $f : U \to \mathbb{C}$ be complex differentiable at every point in the domain. Given $z_0 \in U$ choose $\delta > 0$ sufficiently small that $B(z_0, \delta) \subset U$. Let $\gamma = C(z_0, \delta)$ denote the simple closed curve around z_0 of radius δ which we will keep fixed.

Consider the function on $B(z_0, \delta)$ defined by

$$B(z_0, \delta) \ni z \mapsto f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^2} d\xi \tag{3}$$

and observe that this map is continuous. Writing $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$ we recall that the complex derivative takes the form

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y) = \frac{\partial u}{\partial y}(x, y) - i\frac{\partial v}{\partial y}(x, y)$$

Thus by considering the real and imaginary parts of this expression the continuity of the map in (3) gives the continuity of the the four functions

$$U \ni z \mapsto \frac{\partial u}{\partial x}(z), \frac{\partial v}{\partial x}(z), \frac{\partial u}{\partial y}(z), \frac{\partial v}{\partial y}(z) \in \mathbb{R}$$

as required.

5 Special properties of analytic functions

We want to describe some special properties of analytic functions.

5.1 Maximum Modulus Principle. The following theorem shows that the supremum of the modulus $|f(z)|$ of a non-constant analytic function $f : U \to \mathbb{R}$ isn't achieved inside U.

Theorem 5.1 (Maximum Modulus Principle). *Let $f : U \to \mathbb{C}$ be analytic and let U be connected. If $z_0 \in U$ satisfies $|f(z_0)| \geq |f(z)|$ for all $z \in U$ then $f(z)$ is necessarily a constant function.*

Proof. By multiplying by a constant, if necessary, we can assume without loss of generality that $f(z_0) \in \mathbb{R}^+$. Let us denote

$$\mathcal{S} = \{z \in U : |f(z)| = f(z_0)\} \neq \emptyset.$$

Since f is continuous we see that $\mathcal{S} \subset U$ is a closed set.

We claim that $\mathcal{S}$ is also an open set. To see this, given $\xi \in \mathcal{S} \subset U$ choose $r > 0$ sufficiently small that $B(\xi, r) \subset U$. Then for any $0 < \delta < r$ we have that

$$|f(\xi)| = \left| \int_0^{2\pi} f\left(\xi + \delta e^{2\pi i t}\right) dt \right| \leq \int_0^{2\pi} \left| f\left(\xi + \delta e^{2\pi i t}\right)\right| dt \leq |f(\xi)|$$

where the first equality follows from the Gauss Mean Theorem and the last inequality holds since the integrand $|f(\xi + \delta e^{2\pi i t})| \leq |f(\xi)|$ by assumption. Thus since the first and last terms are the same all of the inequalities must be equalities and so

$$|f(\xi + \delta e^{2\pi i t})| = |f(\xi)| = f(z_0)$$

for all $0 \leq t < 2\pi$ and all $0 < \delta < r$. In particular, for $z \in B(\xi, \delta)$ we have that $|f(z)| = f(z_0)$ from which we can deduce that $\mathcal{S}$ contains the open ball $B(\xi, \delta) \subset \mathcal{S}$. This shows that $\mathcal{S}$ is an open set.

Finally, since $\mathcal{S}$ is both open and closed and U is connected we deduce that $\mathcal{S} = U$ and thus f takes the constant value $f(z_0)$. □

An interesting related result appears in the particular context of analytic functions defined on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

hskip 0.5cm

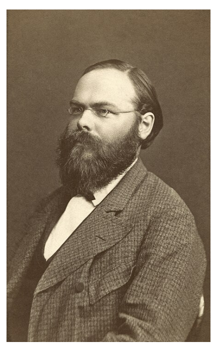


Figure 7: Hermann Schwarz (1843-1921)

Theorem 5.2 (Schwarz Lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic map such that $f(0) = 0$ then*

1. *$|f(z)| \leq |z|$; and*
2. *$|f'(0)| \leq 1$. Moreover, if either*
   - *(a) $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$ or*
   - *(b) $|f'(0)| = 1$*

then f is a "rotation" of the form $f(z) = \rho_\theta(z) := e^{i\theta} z$, for some $0 \leq \theta < 2\pi$.

Proof. We claim that the function $g : \mathbb{D} \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(z) & \text{if } z = 0 \end{cases}$$

is analytic on $\mathbb{D}$. It is clearly analytic for $z \neq 0$ and we see that it is continuous at $z = 0$ since

$$\lim_{|z| \to 0} g(z) = \lim_{|z| \to 0} \frac{f(z)}{z} = f'(0) = g(0).$$

Thus by the removable singularity theorem we have that $g : \mathbb{D} \to \mathbb{C}$ is analytic.

For any $1 > \epsilon > 0$ we can apply the Maximum Modulus Principle to the restriction $g : U \to \mathbb{C}$ where we take

$$U = B(0, 1-\epsilon) = \{z \in \mathbb{C} : |z| \leq 1 - \epsilon\}$$

to deduce that for $|z| \leq 1 - \epsilon$ we have that

$$|g(z)| \leq \sup_{|z| \leq 1-\epsilon} \left| \frac{f(z)}{z} \right| \leq \frac{1}{1-\epsilon}.$$

since $\sup_{|z|\leq 1-\epsilon} |f(z)| \leq 1$. Letting $\epsilon \to 0$ we deduce that $|g(z)| \leq 1$ whenever $|z| \leq 1$. This completes the proof of the first part.

For the second part, we first consider the case that $|f(z_0)| = |z_0|$ for $z_0 \neq 0$ then $|g(z_0)| = 1$. The maximum principle applied to g now tells us that g has a constant value, which we can take to be $e^{i\theta}$, say, for some $0 \leq \theta \leq 2\pi$. This implies that $f(z)$ is a rotation.

Similarly, in the case that $|f'(0)| = 1$ we deduce that $|g(0)| = 1$. The maximum principle applied to g again tells us that g has a constant value, which we can take to be $e^{i\theta}$, say, for some $0 \leq \theta \leq 2\pi$ as before.

□

Example 5.3. Simple examples of analytic functions $f : \mathbb{D} \to \mathbb{D}$ are given by Möbius transformations of the form

$$\phi_a(z) = \frac{z-a}{1-\overline{a}z}$$

where $a \in \mathbb{D}$.

The following result will be useful later.

Proposition 5.4. *Any analytic function $f : \mathbb{D} \to \mathbb{D}$ which is a bijection is of the form $\phi_a \circ \rho_\theta$.*

Proof. We first onserve that ϕ_a extends to the closed disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ that $\phi_a(\partial\overline{\mathbb{D}}) = \partial\overline{\mathbb{D}}$ where

$$\partial\overline{\mathbb{D}} = C(0,1) = \{z \in \mathbb{C} : |z| = 1\}.$$

More precisely, if $|z| = 1$ then

$$|\phi_a(z)| = \left| \frac{z-a}{1-\overline{a}z} \right| = \left| \frac{1}{z} \right| . \left| \frac{z-a}{\overline{z}-\overline{a}} \right| = 1.$$

Observe that $\phi_a^{-1} = \phi_{-a}$, showing that $\phi_a : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a bijection.

Let $f(0) = b$, say. Consider $g = \phi_b \circ f$, which is now an analytic function $g : \mathbb{D} \to \mathbb{D}$ for which $g(0) = 0$ (since $\phi_b(b) = 0$). By the Schwarz Lemma, $|g'(0)| \leq 1$. Similarly, we can apply the Schwarz lemma to $g^{-1} : \mathbb{D} \to \mathbb{D}$ to deduce that $\frac{1}{|g'(0)|} = |(g^{-1})'(0)| \leq 1$. Combining these two inequalities we have that $|g'(0)| = 1$.

Finally, by the uniqueness part of the Schwarz Lemma we have that $g(z) = \rho_\theta$ for some $0 \leq \theta < 2\pi$. But then this means that $f(z) = \phi_{-b} \circ \rho_\theta$. The result follows by taking $a = -b \in \mathbb{D}$. □

5.2 Entire functions We can consider the particular case of analytic functions defined on the entire complex plane.

Definition 5.5. A function $f : \mathbb{C} \to \mathbb{C}$ which is analytic at each point $z \in \mathbb{C}$ is called *entire*.

We begin with another application of the Taylor series version of analyticity.

Theorem 5.6 (Liouville's Theorem)**.** *If an entire function $f : \mathbb{C} \to \mathbb{C}$ is analytic and bounded (i.e., $\sup_{z\in\mathbb{C}} |f(z)| = M < +\infty$) then f is constant.*

Proof. We can write $f(z)$ as a power series representation centred on 0,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with infinite radius of convergence. For any any $R > 0$ and $n \geq 0$ we use an estimate for a_n of the form

$$|a_n| = \left| \frac{1}{2\pi i} \int_{C(0,R)} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{2\pi R}{2\pi} \frac{M}{R^{n+1}} = \frac{M}{R^n}$$

where M is an upper bound for $|f(z)|$ for $z \in \mathbb{C}$, and taking limits as $R \to \infty$ to get $a_n = 0$ except for $n = 0$, i.e., $f(z) = a_0$.

□

We can apply Liouville's theorem to prove a classical result.

Corollary 5.7 (Fundamental Theorem of algebra)**.** *Every non-constant polynomial has a root (i.e., a zero, that is there exists z_0 such that $p(z_0) = 0$).*

Proof. Assume for a contradiction that the polynomial $p(z)$ does not vanish. Then $1/p(z)$ is entire and since $|p(z)| \to +\infty$ as $|z| \to +\infty$ we have that $1/p(z) \to 0$ as $|z| \to +\infty$ and hence for any $\epsilon > 0$ there exists $R > 0$ such that $|1/p(z)| < \epsilon$ for $|z| > R$ and $|1/p(z)|$ is bounded for $|z| \leq R$. In particular, $1/p(z)$ is bounded throughout $\mathbb{C}$ so $1/p(z)$ is constant, contradicting the fact that $p(z)$ is non-constant. □

In particular, this implies that the polynomial can be written as:

$$p(z) = a_n z^n + \cdots + a_0 = a_n(z - z_1)(z - z_2) \cdots (z - z_n).$$

To see this, let z_1 be a zero given by the corollary and then write

$$\begin{aligned} p(z) &= a_n(z - z_1 + z_1)^n + a_{n-1}(z - z_1 + z_1)^{n-1} + \cdots + a_0 \\ &= a_n(z - z_1)^n + a'_{n-1}(z - z_1)^{n-1} + \cdots + a'_1(z - z_1) + a'_0 \end{aligned}$$

Clearly $a'_0 = 0$, by evaluation at z_1. So $p(z) = (z-z_1)q(z)$ where $q(z)$ is a polynomial of degree $n - 1$ (and the leading coefficient of $q(z)$ is 1). Repeating this on $q(z)$ we obtain by induction

$$p(z) = a_n z^n + \cdots + a_0 = a_n(z - z_1)(z - z_2) \cdots (z - z_n)$$

as required.

6 Location of zeros

An interesting problem in complex analysis is getting information on the location of zeros of analytic functions.

6.1 The Argument Principle We begin with a very simple but useful lemma which we will use later.

Lemma 6.1. *Suppose that $f : U \to \mathbb{C}$ is analytic and non-zero then there exists an analytic function $h : U \to \mathbb{C}$ such that $e^{h(z)} = f(z)$ for all $z \in U$.*

Proof. Since f is analytic we have that f' is analytic and since f does not vanish then f'/f is analytic. Therefore, f'/f has a primitive h by integrating f, i.e., if we fix $z_0 \in U$ then for any $z \in U$ we can write $h(z) = \int_{\gamma_{z_0,z}} f(z)dz$ where $\gamma_{z_0,z}$ is (any) path from z_0 to z. [22] In particular, we have that $h' = f'/f$. Thus

$$\frac{d}{dz}\left(f(z)e^{-h(z)}\right) = f'(z)e^{-h(z)} - f(z)h'(z)e^{-h(z)} = 0$$

so $f(z)e^{-h(z)} = e^a \neq 0$ a non-zero constant. Thus we can write $f(z) = e^{a+h(z)}$ □

We can use this lemma to show the following theorem, which counts the number of zeros inside a simple closed curve γ.

Theorem 6.2 (The Argument Principle)**.** *Assume $f : U \to \mathbb{C}$ is analytic and let $\gamma \subset U$ be a simple closed piecewise smooth curve. Assume that f has no zeros on γ. Then the number of zeros $N = N(f, \gamma)$ inside γ is given by*

$$N = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz.$$

Proof. Assume that z_0 is a zero of order $n \geq 1$ for $f(z)$ and z_0 lies inside γ. Assume for simplicity that we have no other zeros inside γ, i.e., $n = N$. Using the lemma above we can then write $f(z) = (z - z_0)^n e^{h(z)}$ and then

$$f'(z) = (z - z_0)^n e^{h(z)}h'(z) + n(z - z_0)^{n-1}e^{h(z)}$$

and therefore

$$\frac{f'(z)}{f(z)} = h'(z) + \frac{n}{z - z_0}$$

Since the first term is analytic we can apply Goursat's theorem to the first term and Cauchy's integral theorem to the second term to write

$$N = n = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz.$$

With more zeroes the argument naturally generalizes. □

The following example essentially re-does the proof above in the special case $N = 1$.

Example 6.3. Assume $f : \mathbb{D} \to \mathbb{C}$ has a simple zero at 0 (i.e., $N = 1$) and that $\gamma = \{z : |z| = r\}$ with $0 < r < 1$. Writing

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \text{ and } f'(z) = \sum_{n=1}^{\infty} n a_{n+1} z^n$$

(with $a_0 = 0$ and $a_1 \neq 0$) we see that

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + g(z)$$

[22] We need to assume that U is simply connected to make h well defined and independent of the path.

where $g(z)$ is analytic on $\mathbb{D}$. By Goursat's theorem $\int_\gamma g(z)dz = 0$ and a direct calculation gives

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{1}{z}dz}_{=1} + \underbrace{\frac{1}{2\pi i}\int_\gamma g(z)dz}_{=0}$$

We mention in passing one of the most important examples of zero counting.

Remark 6.4 (localizing zeros of Riemann Zeta function)**.** One of the major open problems in mathematics is the *Riemann Hypothesis.* For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ we define the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(It converges since $|\zeta(s)| \leq \sum_{n=1}^{\infty} n^{-\mathrm{Re}(s)} < +\infty$). Moreover, it has an analytic extension to $\mathbb{C} - \{1\}$, i.e., there is an analytic function $f : \mathbb{C} - \{1\} \to \mathbb{C}$ such that $f(s) = \zeta(s)$ for $Re(s) > 1$.

Riemann Hypothesis (Conjecture, 1859) The only zeros for $f(s)$ occur at $s = -2, -4, -6, \cdots$ and on the line

$$L := \{s = \frac{1}{2} + it : t \in \mathbb{R}\}.$$

This is Hilbert's 8th problem from his famous list of 23 open problems from 1900, and one of the 7 Millenium problems from 2000 for which the Clay Institute offered a million dollars.

Computer searches for counter-examples (so far unsuccessful!) use the Argument Theorem. One considers a small closed curve γ away from the line L and estimates the integral $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$. One tries to find a γ such that the integral is non-zero.

6.2 Rouché's Theorem Our second approach to locating zeros is particularly useful in relating nearby zeros of nearby functions.

Figure 8: Édouard Rouche (1820-1883)

Theorem 6.5 (Rouche's Theorem)**.** *Assume that $f, g : U \to \mathbb{C}$ are analytic. Let $\gamma \subset U$ be a simple closed curve and assume that neither $f(z)$ or $g(z)$ have zeros on γ and that for each $z \in \gamma$ we have that $|g(z)| < |f(z)|$. Then f and $f + g$ have precisely the same number of zeros inside γ.*

Proof. For each $0 \leq t \leq 1$ we can consider the analytic function $F_t : U \to \mathbb{C}$ defined by $F(z) = F_t(z) := f(z) + tg(z)$.

We claim that for any $0 < t \leq 1$ the function $F_t(z)$ also has no zeros on γ. To see this, assume for a contradiction that for some $z \in \gamma$ we have $F_t(z) = f(z)+tg(z) = 0$ then $|f(z)| = t|g(z)| < |f(z)|$, by the hypothesis on g in the statement.

Let N_{F_t} denote the number of zeros for $F_t(z)$ inside the curve γ. In particular, $N_{F_0} = N_f$ and $N_{F_1} = N_{f+g}$ are the number of zeros inside γ for each of the two functions f and $f + g$, respectively.

We can use the argument principle to write

$$N_{F_t} = \frac{1}{2\pi i} \int_\gamma \left(\frac{F_t'(z)}{F_t(z)} \right) dz \in \mathbb{Z}^+$$

which is clearly continuous in $t \in [0, 1]$ from the definition. Therefore, for $t > 0$ sufficiently small we have that $N_{F_t} = N_f$. But since the function is valued in $\mathbb{Z}_+$ and continuous we have $N_{f+g} = N_{F_1} = N_{F_0} = N_f$. In particular, this means that $N_{f+g} = N_f$ as required. □

Example 6.6. Show that all four of the zeros of $z^4 - 7z - 1$ lie in the ball $B(0, 2) = \{z \in \mathbb{C} : |z| < 2\}$.

Let $f(z) = z^4$ then all four zeros lie inside the ball $B(0, 2) = \{z \in \mathbb{C} : |z| < 1\}$. In particular, $z = 0$ is a zero of multiplicity four. Now set $g(z) = -7z - 1$ and let $\gamma = C(0, 2) = \{z \in \mathbb{C} : |z| = 2\}$. Whenever $|z| = 2$ we have that

$$|g(z)| = | - 7z - 1| \leq 7|z| + 1 = 15 < 16 = |z^4| = |f(z)|.$$

and thus for $z \in \gamma$ we have $|g(z)| < |f(z)|$. Thus by Rouché's theorem we have that $f(z) + g(z) = z^4 - 7z - 1$ has the same number of zeros in $B(0, 2)$ as $f(z)$, i.e., four.

7 Sequences of analytic functions

Rather than a single analytic function we can consider familes $\{f_n : U \to \mathbb{C}\}$ of such functions. The following two results deal with such sequences.

7.1 Weierstrass-Hurwitz Theorem The first result deals with the limit when we assume we have a convergent squence of analytic functions.

Figure 9: Adolf Hurwitz (1859-1919)

Theorem 7.1 (Weierstrass-Hurwitz Theorem)**.** *Assume $f_n : U \to \mathbb{C}$ $(n \geq 1)$ are a sequence of non-zero analytic functions. If f_n converges uniformly on compact sets to a continuous function f (i.e., if $K \subset U$ is compact then $\sup_{z\in K} |f(z) - f_n(z)| \to 0$ as $n \to +\infty$) then:*

1. *$f : U \to \mathbb{C}$ is analytic*
2. *f either has no zeros or is identically zero.*

Proof. For the first part (due to Weierstrass) one only needs to show that for any simple closed curve $\gamma \subset U$ we have that

$$\left|\int_\gamma f(z)dz\right| \leq \underbrace{\left|\int_\gamma f_n(z)dz\right|}_{=0} + \left|\int_\gamma (f(z) - f_n(z))dz\right|$$

$$\leq 0 + \text{length}(\gamma) \underbrace{\left(\sup_{z\in K} |f(z) - f_n(z_)|\right)}_{\to 0}$$

where the first term is zero by Goursat's theorem (since $f_n : U \to \mathbb{C}$ is analytic). Thus $\int_\gamma f(z)dz = 0$ for all simple closed curves, and so f is analytic by Moreira's Theorem.

For the second part (due to Hurwitz), if $f(z)$ isn't identically zero then the set of zeros $\{z_i\}_{i=1}^N$ for $f(z)$ is finite. We need to show that this set is empty. Assume for a contradiction it isn't empty and let γ be a closed simple curve which contains one of these zeros z_1, say. By the argument principle

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{f_n'(z)}{f_n(z)}dz \to \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz > 0$$

as $n \to +\infty$ which gives a contradiction. □

7.2 An application to the Riemann zeta function Recall that the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$$

was claimed to analytic for $Res(s) > 1$. To check this we can observe that

$$f_N(s) = \sum_{n=1}^N \frac{1}{n^s} \left(= \sum_{n=1}^N \exp(-s\log n)\right)$$

is analytic on $\mathbb{C}$. For any compact set $K \subset U := \{s \in \mathbb{C} : Re(s) > 1\}$, say, we have that $\sup_{s\in K} |\zeta(s) - f_n(s)| \to 0$. Thus by the theorem $\zeta(s)$ is analytic on U.

7.3 Montel's Theorem. Recall that a set $K \subset \mathbb{C}$ is compact if and only if it is closed and bounded. We begin with a result on families of continuous functions.

Definition 7.2. Let $\{f_n\}_{n=1}^\infty$ be a family of continuous functions $f_n : U \to \mathbb{C}$ for $n \geq 1$. We say that this family is *normal* if there is a subsequence $\{f_{n_k}\}_{k=1}^\infty$ which converges uniformly on every compact subset of U to a continuous function $f : U \to \mathbb{C}$, i.e., for every compact subset $K \subset U$ we have that $\sup_{z\in K} |f_{n_i}(z) - f(z)| \to 0$ as $n_i \to +\infty$.

In the case of analytic functions, we can get that the family is normal by just assuming the sequence is uniformly bounded.

Figure 10: Paul Montel (1876-1975)

Theorem 7.3 (Montel's Theorem)**.** *Let $f_n : U \to \mathbb{C}$, $n \geq 1$, be a family of analytic functions. Assume that there is a positive constant $M > 0$ such that*

$$|f_n(z)| \leq M \text{ for all } z \in U, \text{ and all } n \geq 1$$

then $\{f_n\}_{n=1}^{\infty}$ is a normal family. Moreover, the limiting function $f : U \to \mathbb{C}$ is analytic.

The proof is based on a classical result in real analysis. We recall the following definition.

Definition 7.4. Let K be a compact set. Let $\{f_n\}_{n=1}^{\infty}$ be a family of continuous functions $f_n : K \to \mathbb{C}$, for $n \geq 1$. We say that this family is *equicontinuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $z, w \in K$ with $|z - w| < \delta$ then $|f_n(z) - f_n(w)| < \epsilon$ for all $n \geq 1$.

The corresponding result from metric spaces we need is the following.

Theorem 7.5 (Arzela-Ascoli)**.** *Any equicontinuous family $f_n : K \to \mathbb{C}$ has a subsequence (f_{n_k}) which is uniformly convergent to a continuous function $f : K \to \mathbb{C}$ (i.e., $\sup_{z \in K} |f_{n_k}(z) - f(z)| \to 0$ as $n_k \to +\infty$)*

We omit the proof of this result.

Proof of Montel's theorem. It suffices to show that the Arzela-Ascoli applied to the restrictions $f_n : K \to \mathbb{C}$ for any one-empty closed set $K \subset U$.

Let $z_0 \in U$ and let $\epsilon > 0$.

Let $r > 0$ be sufficiently small that $\overline{B(z_0, 2r)} \subset U$. We have that the restrictions $f_n : \overline{B(z_0, 2r)} \to \mathbb{C}$ have the bounds

$$\sup_{z \in \overline{B(z_0, 2r)}} |f(z)| \leq M.$$

It then follows from Cauchy's Theorem that for all $z, w \in B_{2r}(z_0)$

$$\begin{aligned} f(z) - f(w) &= \frac{1}{2\pi i} \int_{|z-\xi|=2r} \left(\frac{f_n(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int_{|z-\xi|=2r} \frac{f_n(\xi)}{\xi - w} d\xi \right) \\ &= \frac{z - w}{2\pi i} \int_{|z_0 - z|=2r} \frac{f_n(\xi)}{(\xi - z)(\xi - w)} d\xi \end{aligned}$$

Thus for any $z, w \in \overline{B(z_0, r)}$ we have that

$$|(\xi - z)(\xi - w)| > r^2 \text{ for all } \xi \text{ with } |\xi - z| = 2r.$$

We can therefore bound

$$|f(z) - f(w)| \leq |z - w| \frac{2M}{r}.$$

Therefore with $\delta = \min\{r, r\epsilon/(4M)\}$ we have that wherever $z, w \in B_\delta(z_0)$ we have $|f(z) - f(w)| < \epsilon$ In particular, we see that the restrictions $f_n : \overline{B(z_0, r)} \to \mathbb{C}$, $n \geq 1$, form an equicontinuous family.

If $K \subset U$ is compact we can cover it be finitely many such balls and we again deduce that $f_n : K \to \mathbb{C}$, $n \geq 1$, forms an equicontinuous family. We can then invoke the Arzela-Ascoli theorem to deduce that there is a convergent subsequence $f_{n_k} : K \to \mathbb{C}$, for $k \geq 1$.

The analyticity of the function $f : U \to \mathbb{C}$ follows from the Hurwitz-Weiestrauss theorem.

□

8 Riemann Mapping Theorem

One of the most elegant theorems in complex analysis is the following.

Theorem 8.1 (Riemann Mapping Theorem). *Let $U \subset \mathbb{C}$ be an open subset homeomorphic to $\mathbb{D}$. Then there exists an analytic bijection $f : U \to \mathbb{D}$. Moreover, if we fix $z_0 \in U$ we can also assume that $f(z_0) = 0$ and $f'(z_0) > 0$.*

We will present a fairly complete sketch of the proof. **(This is for entertainment. The proof is non-examinable)**

Proof. The proof is quite elaborate, so we break it down into steps. We assume for simplicity that U is bounded, i.e., $U \subset B(z_0, R)$ for some $R > 0$.

Step 1: Fix $z_0 \in U$ and let $\mathcal{F}$ be the family of those functions $f : U \to \mathbb{D}$ such that

1. f is analytic; and
2. $f(z_0) = 0$.

To show that $\mathcal{F} \neq \emptyset$ we can construct $f : U \to \mathbb{D}$ by

$$f(z) = \frac{(z - z_0)}{2R}$$

since

$$|f(z)| = \left| \frac{(z - z_0)}{2R} \right| \leq \frac{|z| + |z_0|}{2R} \leq \frac{R + R}{2R} \leq 1$$

and by construction we see that $f(z_0) = 0$ and thus $f \in \mathcal{F}$.

Step 2: Since the functions in $\mathcal{F}$ are analytic and bounded (i.e., for $f \in \mathcal{F}$ and $|f(z)| \leq 1$) we can apply Montel's theorem to deduce that the family is normal.

Step 3: We define

$$M = \sup\{|f'(z_0)| : f \in \mathcal{F}\}.$$

Claim: We want to show that this is finite (i.e., $M < +\infty$).

Proof of Claim: Choose $r > 0$ sufficiently small that $\overline{B(z_0, r)} \subset U$. By Cauchy's theorem we have that for any $f \in \mathcal{F}$,

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f(z)}{(z - z_0)^2} dz \right| \leq \frac{1}{r} \underbrace{\left(\sup_{z \in \mathbb{D}} |f(z)| \right)}_{\leq 1} \leq \frac{1}{r}.$$

Thus the result follows with $M \leq \frac{1}{r}$.

Step 4: We now want to show the supremum is realized.

Claim: There exists $f_0 \in \mathcal{F}$ such that $f_0'(z_0) = M$.

Proof of Claim: We can chose $f_n \in \mathcal{F}$, $n \geq 1$, such that $|f_n'(z_0)| \to M$ as $n \to +\infty$. By Montel's Theorem applied to $\{f_n\}$ we can choose a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ and an analytic function $f : U \to \mathbb{C}$ such that $f_{n_k} \to f$ uniformly on any compact set $K \subset \mathbb{C}$. Therefore, we see that $|f'(z_0)| = M$.

If we replace $f(z)$ by $e^{i\theta} f(z)$ then we can choose $0 \leq \theta < 2\pi$ so that $f'(z_0) = M$.

Step 5: We want to show that $f : U \to \mathbb{D}$ is injective.

Claim: $f : U \to \mathbb{D}$ is injective.

Proof of Claim: Let $z_1 \neq z_2$ be distinct points in U. Let $0 < \rho = |z_1 - z_2|$ and define $g_k : \overline{B(z_2, \rho)} \to \mathbb{C}$ by

$$g_k(z) = f_{n_k}(z) - f_{n_k}(z_1)$$

Since the functions $f_{n_k} (\in \mathcal{F})$ are all injective, the functions g_k are all non-zero on $\overline{B(z_2, \rho)}$ (i.e., $\nexists z \in \overline{B(z_2, \rho)}$ with $f(z) = 0$).

By the Hurwitz-Weierstrauss theorem the limit function $g(z) = f(z) - f(z_1)$ is either identically zero or has no zeros. But, it cannot be identically zero (since we assume that $|f'(z_0)| = M > 0$). Thus $f(z) \neq f(z_1)$ for all $z \in \overline{B(z_2, \rho)}$ and thus $f(z_2) \neq f(z_1)$.

Step 6: We want to show that $f : U \to \mathbb{D}$ is surjective.

Claim: $f : U \to \mathbb{D}$ is surjective.

Proof of Claim: Assume for a contradiction that f is not surjective, and thus there exists $w \in \mathbb{D}$ which is not in the image $f(U)$.

We can choose a Möbius map $\phi_w : \mathbb{D} \to \mathbb{D}$ by

$$\phi_w(z) = \frac{z - w}{1 - z\overline{w}}$$

which maps w to 0. We then consider $\phi_w \circ f : U \to \mathbb{D}$. This map now doesn't have 0 in its image. As we saw in a previous lemma, we can write $\phi_w \circ f(z) = e^{k(z)}$ where $k : U \to \mathbb{C}$ is analytic, and thus we can define $g : U \to \mathbb{C}$ by

$$g(z) = (\phi_w \circ f(z))^{1/2} (=: e^{k(z)/2}).$$

In particular, $g : U \to \mathbb{D}$ is one-to-one. Finally, we introduce

$$\phi_{g(z_0)}(z) = \frac{z - g(z_0)}{1 - z\overline{g(z_0)}}$$

and define $\rho : U \to \mathbb{D}$ by $\rho(z) = \phi_{g(z_0)} \circ g(z_0)$.

We can then compute

$$\rho'(z_0) = \underbrace{\phi_{g(z_0)}'(g(z_0))}_{=\frac{1}{1-|g(z_0)|^2}} . g'(z_0) \tag{1}$$

and

$$\begin{aligned} \left(g(z_0)^2\right)' &= 2g'(z_0).g(z_0) \\ &= \phi_w'(f(z_0)).f'(z_0) \\ &= (1 - |w|^2) f'(z_0). \end{aligned} \tag{2}$$

(after a little calculation).)

From (1) and (2) we find that:

$$\begin{aligned}\rho'(z_0) &= \frac{1}{1-|g(z_0)|^2}\left(\frac{1-|w|^2}{2g(z_0)}\right) f'(z_0)\\ &= \frac{1}{1-|w|}\left(\frac{1-|w|^2}{2g(z_0)}\right) f'(z_0)\\ &= \left(\frac{1+|w|}{2g(z_0)}\right) f'(z_0)\end{aligned}$$

However, since $w \neq 0$ then $1+|w| > 1$ and $g(z_0) = \sqrt{|w|}$ and thus

$$\rho'(z_0) = \underbrace{\left(\frac{1+|w|}{2\sqrt{|w|}}\right)}_{>1} f'(z_0)$$

giving $|\rho'(z_0)| > M$, a contradiction to the definition of M.

However, this contradicts $h(z)$ having maximum derivative M at z_0. □

Remark 8.2. If U is unbounded then we can apply a more elaborate argument to prove Step 1. Let $z_0 \in \mathbb{C} - U$. The function $\phi : U \to \mathbb{C}$ is non-vanishing and we can write $\phi(z) = e^{k(z)} = k(z)^2$, where $k(z) = e^{k(z)/2}$. Moreover, $k(z)$ is one-to-one (since $\phi(z)$ is) and there are not distinct points z_1, z_2 such that $h(z_1) = h(z_2)$ (since then $\phi(z_1) = \phi(z_2)$). This is an open mapping and so we can choose $B(b,r) \subset h(U)$. But then $B(-b,r) \cap h(U) = \emptyset$.

We may therefore define the holomorphic function

$$f(z) = \frac{r}{2(h(z)+b)}.$$

Since $|h(z) - h(-z)| \geq r$ for $z \in U$, it follows that $f : U \to \mathbb{D}$. Since h is one-one then so is f. Compositing f with a Möbius map that preserves $\mathbb{D}$ we can get a function which is analytic and one-on-one and bounded by 1.

9 The open mapping theorem and its applications

Another useful property of analytic maps is that analytic maps map open sets to open sets. This particularly useful in giving alternative proofs of results.

Remark 9.1. This is in contrast to the situation on the real line $\mathbb{R}$ where the map $x \mapsto x^2$ takes the open interval $(-1,1)$ to the half open interval $[0,1)$.

9.1 The open mapping theorem. We begin with the statement.

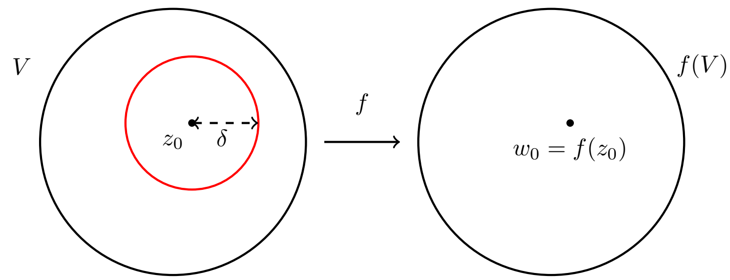
Theorem 9.2 (Open mapping theorem)**.** *Let $f : U \to \mathbb{C}$ be a non-constant analytic map. Then for any non-empty set $V \subset U$ the image $f(V) \subset \mathbb{C}$ is an open set.*

Proof. We want to show that $f(V)$ is an open set, i.e., every point in $f(V)$ is the centre of an some ball which is also contained in $f(V)$

Step 1. Let $w_0 \in f(V)$. Choose $z_0 \in V$ with $f(z_0) = w_0$, i.e., a preimage of w_0. Since V is open we can choose $\delta > 0$ to be sufficiently small that $B := B(z_0, \delta) \subset V$.

We can then define an analytic function $g : B(z_0, \delta) \to \mathbb{C}$ by

$$g(z) = f(z) - w_0$$

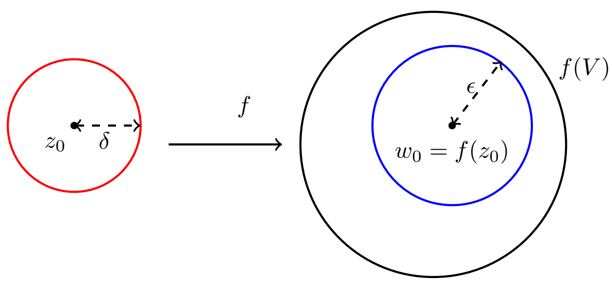


then we see that $g(z_0) = 0$. Moreover, z_0 is an isolated zero for $g(z)$ by the identity theorem. By choosing $\delta > 0$ even smaller (if necessary) we can assume that z_0 is the only zero for $g(z)$ in $\overline{B(z_0, \delta)}$. By compactness of $C(z_0, \delta) = \{z \in \mathbb{C} : |z - z_0| = \delta\}$ we can to deduce that

$$\epsilon = \min_{z \in C(z_0, \delta)} |g(z)| > 0.$$

Step 2. Let us denote $D = B(w_0, \epsilon) \subset \mathbb{C}$. Given any $w_1 \in D$ we can consider the function $h : B \to \mathbb{C}$ defined by

$$h(z) = f(z) - w_1.$$

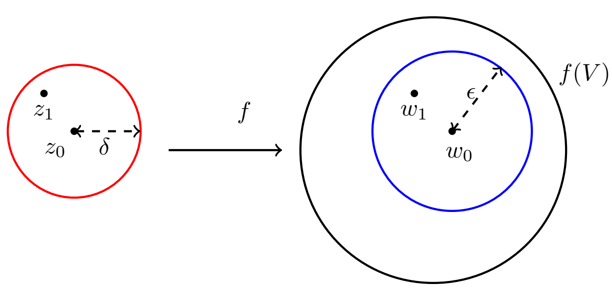


Thus $h(z) = g(z) + (w_0 - w_1)$ and so

$$|h(z) - g(z)| \leq |w_0 - w_1| < \epsilon.$$

Moreover, for $z \in C(z_0, \delta) = \partial B$ we have that $|g(z)| \geq \epsilon$ by the definition of $\epsilon > 0$.

Step 3. Thus by Rouché's theorem $h : B(z_0, \delta) \to \mathbb{C}$ has the same number of zeros as $g : B(z_0, \delta) \to \mathbb{C}$, i.e., precisely one which we denote by z_1 (i.e., $h(z_1) = 0$).



Thus for every $w_1 \in D = B(w_0, \epsilon)$ we can choose a

$$z_1 \in B = B(z_0, \delta) \subset U$$

with $f(z_1) = w_1$. In particular, $D = B(w_0, \epsilon) \subset f(V)$ as required. This completes the proof. $\square$

9.2 Alternative proofs using the Open Mapping Theorem One can use the Open Mapping Theorem to give alternative proofs of some earlier results in the notes. We will illustrate this by giving alternative proofs of the Maximum Modulus Principle and the Fundamental Theorem of Algebra.

9.2.1 Alternative proof of the Maximum Modulus principle.

We want to show that if $f : U \to \mathbb{C}$ is a non-constant analytic function then for any $z_0 \in U$ there exists $z \in U$ with $|f(z_0)| < |f(z)|$.

Suppose for a contradiction that $f : U \to \mathbb{C}$ has that the absolute value $|f(z)|$ attains a maximum at $z_0 \in U$. By the open mapping theorem the image of a neighbourhood V of z_0 is a neighbourhood in $f(V)$ of $f(z_0) \in \mathbb{C}$ and thus will contain points whose absolute value will be larger than $|f(z_0)|$. This contradiction gives the proof.

9.2.2 Alternative proof of the Fundamental Theorem of Algebra.

We want to show that if $f : \mathbb{C} \to \mathbb{C}$ is a non-constant polynomial then there exists $z_0 \in \mathbb{C}$ with $f(z_0) = 0$.

If we assume for a contradiction that $f(z)$ has no zeros then the reciprocal function $1/f : \mathbb{C} \to \mathbb{C}$ defined by $(1/f)(z) := 1/f(z)$ $(z \in \mathbb{C})$ is well defined and analytic. We can now note two inequalities

(i) Since $f(0) \neq 0$ we can choose $R > 0$ sufficiently large that for $|z| > R$ we can bound $|1/f(z)| < 1/(2|f(0)|)$ (since $f(z) = a_n z^n + \cdots a_1 z + a_0$ is a polynomial with $a_n \neq 0$ for some $n \geq 1$).

(ii) Since $\overline{B(0,R)} = \{z \in \mathbb{C} : |z| \leq R\}$ is compact we can choose a maximum value, i.e., there exists $z_0 \in \overline{B(0,R)}$ such that $|1/f(z)| \leq |1/f(z_0)|$ for all $z \in \overline{B(0,R)}$ (and, in particular, $|1/f(0)| \leq |1/f(z_0)|$).

Comparing the inequalities in (i) and (ii) we have:

$$\frac{1}{|f(z)|} \leq \frac{1}{|f(z_0)|} \text{ for all } z \in \mathbb{C}. \tag{1}$$

But by the Open Mapping Theorem the image of any neighbourhood $V \ni z_0$ under $1/f(z)$ is open and so there must be a point $z_1 \in V$ with $1/|f(z_1)| > 1/|f(z_0)|$, contradictng (1).