MA 3B8

THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: April 2026

COMPLEX ANALYSIS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER ALL 4 QUESTIONS.

Each question is 25 marks; the numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. Suppose that z ranges over $\mathbb{C}^\times = \mathbb{C} - \{0\}$. Suppose that x, y, r, and θ range over the reals. Suppose that the variables satisfy the following:

$$z = x + iy \qquad r^2 = x^2 + y^2 \qquad x = r\cos(\theta) \qquad y = r\sin(\theta)$$

a) Derive expressions, valid in $\mathbb{C}^\times$, for dz, dr, and $d\theta$ in terms of x, y, dx, and dy. **[5]**

b) Using the above prove that $\frac{dz}{z} = \frac{dr}{r} + i\,d\theta$. **[4]**

Suppose that $D \subset \mathbb{C}$ is the *diamond*: the convex hull of the points 1, i, -1, and $-i$. (That is, the rotated square with vertices at those points.)

c) Sketch D. In your sketch, include and label the following: the real and imaginary axes, the vertices of D, and the sides $\gamma_k = [i^k, i^{k+1}]$ of D, for $k = 0, 1, 2, 3$. Additionally orient (anticlockwise) the sides γ_k. **[5]**

d) For each $k = 0, 1, 2, 3$ write the side γ_k as an affine function from $[0, 1]$ to $\mathbb{C}$ (with the given orientation). **[3]**

For each $k = 0, 1, 2, 3$, and with γ_k as above, define the contour integral $I_k = \int_{\gamma_k} \frac{dz}{z}$.

e) Using the above or otherwise, for each $k = 0, 1, 2, 3$, compute I_k. **[5]**

f) Using the above or otherwise compute $I = \int_{\partial D} \frac{dz}{z}$. **[3]**

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2. Suppose that f is a function from $\mathbb{C}$ to $\mathbb{C}$.

a) Define what it means for f to be *holomorphic*. **[2]**

b) Prove that constant functions, and the identity, are holomorphic. **[3]**

Suppose now that $f, g\colon \mathbb{C} \to \mathbb{C}$ are holomorphic.

c) Prove that $f + g$ and $f \cdot g$ are holomorphic. **[5]**

Suppose now that $f\colon \mathbb{C} \to \mathbb{C}$ is a function.

d) Prove that $f(z) = \bar{z}$ is not holomorphic. **[6]**

Suppose that $z = x + iy \in \mathbb{C}$; here x and y are the real and imaginary parts of z. Suppose now that $f, g, h\colon \mathbb{C} \to \mathbb{C}$ are functions as defined below. For each, decide if it is holomorphic or not; give brief justifications.

e) $f(z) = x$ **[3]**

f) $g(z) = x^2 - y^2 + 2ixy$ **[3]**

g) $h(z) = x^2 + y^2$ **[3]**

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3. a) State the residue theorem. **[5]**

Consider the following pair of improper definite integrals:

$$I = \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx \qquad\qquad J = \int_{0}^{\infty} \frac{\sqrt{x}}{1+x^2}\,dx$$

b) For each, determine if it converges. **[4]**

For each integral that does converge, compute it as follows.

c) Choose an appropriate meromorphic function. **[2]**

d) Choose and sketch a useful contour. **[4]**

e) Compute residues. **[3]**

f) Identify the arcs where the integral tends to zero. **[3]**

g) Integrate along the arcs where the integral does not tend to zero. **[2]**

h) Take limits and apply the residue theorem. **[2]**

(You do not need to make any numerical estimates to double-check your work.)

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4. a) Suppose that $T = \{\infty, 0, 1\} \subset \hat{\mathbb{C}}$.

(i) Find all conformal automorphisms of $\hat{\mathbb{C}}$ that fix T setwise. **[6]**

(ii) Briefly justify your answer. **[3]**

b) We define the domain W as follows:

$$W = \{z \in \mathbb{C} : |z| < 1,\ 0 < \textsc{Imag}(z) < \textsc{Real}(z)\}$$

By the Riemann mapping theorem, the domain W is conformally equivalent to $\mathbb{D}$, the open unit disc. Give a direct proof of this by finding a finite sequence of explicit biholomorphic mappings along a finite sequence of domains in the complex plane, starting with W and ending with $\mathbb{D}$. For each mapping write out the mapping and its inverse. Also, for each mapping sketch its domain and its image. In each sketch include and label the real and imaginary axes; label points of interest in the domain as well as their images.

(You do not need to justify your work. You do not need to produce the overall composition of biholomorphic mappings. The sixteen marks will be split as evenly as possible between the sketches.) **[16]**

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