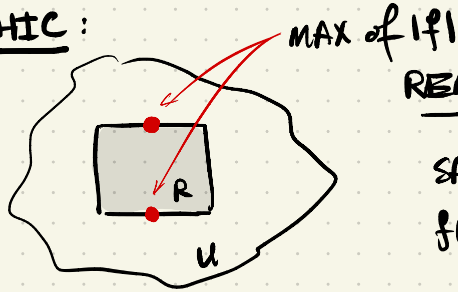
LECTURE 23 2025-11-28 SAUL SCHLEIMER MA3B8

① THE MAXIMUM MODULUS PRINCIPLE

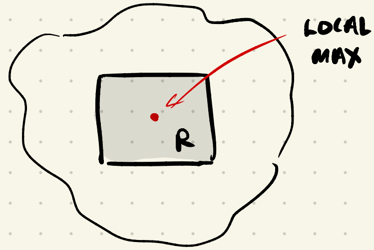
THM: SUPPOSE U IS A DOMAIN. SUPPOSE $R \subset U$ IS A REGION [COMPACT, NICE BOUNDARY]. SUPPOSE $f: U \to \mathbb{C}$ HOLOMORPHIC. IF $\max\{|f(z)| : z \in R\}$ IS REALISED IN THE INTERIOR of R THEN f IS CONSTANT.

HOLOMORPHIC:



REAL ANALYTIC

SAY FOR $f(x,y) = -x^2 - y^2$



PROOF: SUPPOSE $z_0 \in R - \partial R$ REALISES THE MAX of $|f|$ ON R. SET $M_0 = |f(z_0)|$. FIX $r > 0$ SO THAT $B = B(z_0; r) \subset R$. MEAN VALUE THM GIVES

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

SO: $$M_0 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} M_0\, d\theta = M_0.$$

SO: $$\frac{1}{2\pi} \int_0^{2\pi} M_0\, d\theta = M_0 = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta.$$

SO: $$\frac{1}{2\pi} \int_0^{2\pi} [\underbrace{M_0 - |f(z_0 + re^{i\theta})|}_{\text{NON-NEGATIVE}}]\, d\theta = 0.$$

BUT $M_0 - |f(z_0+re^{i\theta})| \geq 0$ FOR ALL θ.

THUS $M_0 = |f(z_0+re^{i\theta})|$ FOR ALL θ.

NOTE: IF $M_0 = 0$ THEN $f(z_0+re^{i\theta}) = 0$ FOR ALL θ.

THEN $f(z) \equiv 0$ BY IDENTITY THM. DONE ✓

SUPPOSE INSTEAD THAT $M_0 \neq 0$. SO $f(z_0) \neq 0$.

DEFINE $g: U \to \mathbb{C}$ BY $g(z) = f(z)/f(z_0)$. SO $g(z_0) = 1$.

DEFINE
$$\left.\begin{aligned} c(\theta) &= \text{REAL}\left(g(z_0+re^{i\theta})\right) \\ s(\theta) &= \text{IMAG}\left(g(z_0+re^{i\theta})\right) \end{aligned}\right\} \text{ CONTINUOUS AND } c^2+s^2=1.$$

SO: MVT GIVES

$$1 = g(z_0) = \frac{1}{2\pi}\int_0^{2\pi} g(z_0+re^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left(c(\theta)+i\,s(\theta)\right)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} c(\theta)\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} s(\theta)\,d\theta$$

SO: $1 = \frac{1}{2\pi}\int_0^{2\pi} c(\theta)\,d\theta$, $0 = \frac{1}{2\pi}\int_0^{2\pi} s(\theta)\,d\theta$.

NOTE $c(\theta) \leq 1$ SO $\frac{1}{2\pi}\int_0^{2\pi}\underbrace{\left(1-c(\theta)\right)}_{\text{NON-NEG}}d\theta = 0$ SO $c(\theta) \equiv 1$. SO $s(\theta) \equiv 0$.

SO g CONSTANT ON $\partial B(z_0; r)$ SO CONSTANT (BY ID. THM).

SO $f(z) \equiv f(z_0)$ IS CONSTANT. □

② THE MINIMUM MODULUS PRINCIPLE

THM: SUPPOSE U IS A DOMAIN, $f: U \to \mathbb{C}$ HOLOMORPHIC.

SUPPOSE $z_0 \in U$ IS A LOCAL MIN FOR $|f|: U \to \mathbb{R}$.

IF $f(z_0) \neq 0$ THEN f IS CONSTANT.

[EQUIV: IF f NONCONSTANT THEN $f(z_0) = 0$.]

PROOF: SUPPOSE $f(z_0) \neq 0$. FIX $r > 0$ SO THAT

(1) $\overline{B(z_0;r)} \subset U$

(2) $\overline{B(z_0;r)} \cap Z(f) = \emptyset$.

LET V BE OPEN WITH

(1) $\overline{B(z_0;r)} \subset V \subset U$

(2) $V \cap Z(f) = \emptyset$. [$\overline{B(z_0;r)}$ COMPACT ETC].

DEFINE $g : V \to \mathbb{C}$ BY $g(z) = 1/f(z)$, SO g HOLOMORPHIC.

SO z_0 IS LOCAL MAX FOR $g(z)$, SO g CONSTANT, SO f CONSTANT. $\square$

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③ <u>OPEN MAPPING THEOREM</u> [SUPER FALSE OVER $\mathbb{R}$: EG $x \mapsto x^2$]

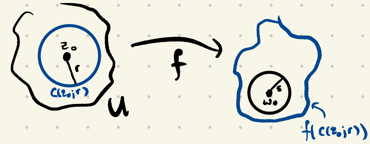
<u>THM</u>: SUPPOSE $f : U \to \mathbb{C}$ HOLOMORPHIC, NON-CONSTANT. THEN $f(U)$ IS OPEN.

[<u>COROLLARY</u>: $f(U)$ IS A DOMAIN]

<u>PROOF</u>: FIX $z_0 \in U$. DEFINE $w_0 = f(z_0)$ AND $g(z) = f(z) - w_0$.
SINCE f NONCONSTANT, g IS NONCONSTANT. SO $Z(g)$ IS ISOLATED IN U. PICK $r > 0$ SO THAT

(1) $\overline{B(z_0;r)} \subset U$

(2) $\overline{B(z_0;r)} \cap Z(g) = \{z_0\}$.



DEFINE: $2\varepsilon = \text{MIN}\{ |g(z)| : z \in C(z_0;r) \}$. SO $\varepsilon > 0$.

<u>CLAIM</u>: $B(w_0;\varepsilon) \subset f(B(z_0;r))$

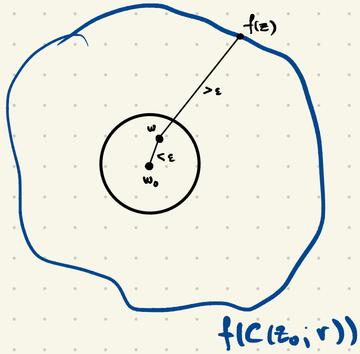
<u>PROOF</u>: FIX $w \in B(w_0;\varepsilon)$.

DEFINE $g_w : U \to \mathbb{C}$ BY $g_w(z) = f(z) - w$.

SO $|g_w(z_0)| = |f(z_0) - w| = |w_0 - w| < \varepsilon$.

SUPPOSE $z \in C(z_0;r)$. THEN WE HAVE:

$$
\begin{aligned}
|g_w(z)| &= |f(z) - w| \\
&= |f(z) - w_0 + w_0 - w| \\
&\geq |f(z) - w_0| - |w_0 - w| \\
&\geq |g(z)| - |w_0 - w| \\
&> 2\varepsilon - \varepsilon = \varepsilon \\
&> |g_w(z_0)|
\end{aligned}
$$



<u>THUS</u>: $\min\{|g_w(z)| : z \in C(z_0;r)\} > |g_w(z_0)|$.

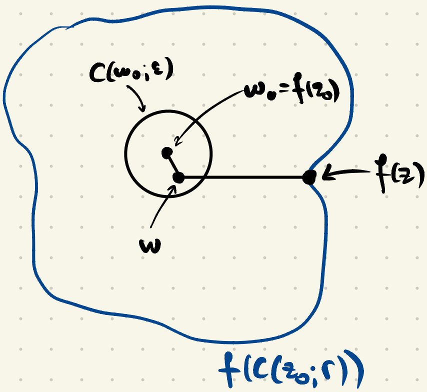
<u>EXERCISE</u>: $|g_w|$ HAS LOCAL MIN IN $B(z_0;r)$.

<u>MIN MOD PRIN</u>: g_w HAS A ROOT IN $B(z_0;r)$.

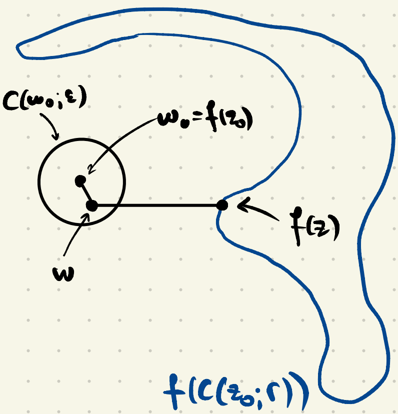
SO SOME $z_1 \in B(z_0;r)$ HAS $g_w(z_1) = 0$. SO $f(z_1) = w$. ◻

THE CLAIM PROVES THE THEOREM ◻.

<u>WHAT HAPPENS</u>



<u>WHAT DOES NOT</u>



<u>MORALLY</u>: THE IMAGE of $C(z_0;r)$ WINDS AROUND THE IMAGE of z_0 (AND THUS ALL POINTS CLOSE TO THE IMAGE.)

<u>SKIPPED.</u>