2025-11-26 LECTURE 22 MA3B8 SAUL SCHLEIMER.

(1) A LOGARITHMIC INTEGRAL

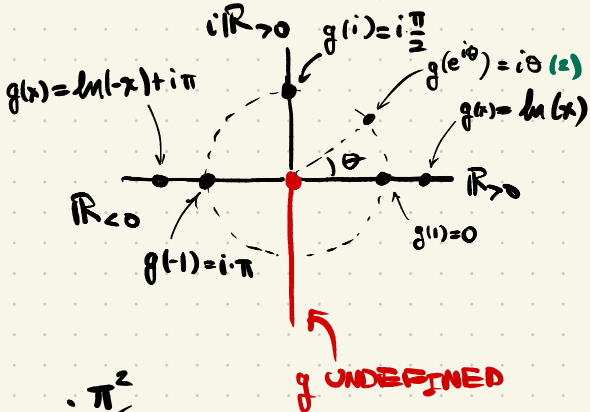
DEFINE: $$J = \int_0^\infty \frac{\ln^2(x)}{1+x^2}\,dx$$

LET $U = \mathbb{C} - R(3\pi/2)$ BE THE CUT PLANE. LET $g : U \to \mathbb{C}$ BE THE BRANCH of LOG WITH

$$g(z) = \log(|z|) + i\arg(z)$$

WHERE (1) $z = |z|e^{i\arg(z)}$

AND (2) $\arg(z) \in (-\pi/2, 3\pi/2)$

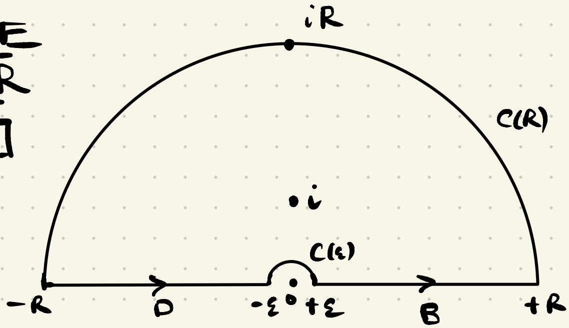


SO: $g(i) = \frac{\pi}{2} i$

DEFINE $h(z) = \dfrac{g^2(z)}{1+z^2}$.

SO $\mathrm{RES}(h, i) = \left(\frac{\pi}{2} i\right)^2 / 2i = \dfrac{-\pi^2/4}{2i} = i \cdot \pi^2/8$

KEYHOLE CONTOUR [SORT of]



DEFINE $A = B \cup C(R) \cup D \cup C(\varepsilon)$ AS SHOWN

$B = [\varepsilon, R]$

$D = [-R, -\varepsilon]$

$C(\varepsilon)$, $C(R)$ UPPER HALVES of $C(0,\varepsilon)$, $C(0,R)$.

SO: $$\int_A h\,dz = 2\pi i \cdot \mathrm{RES}(h, i)$$
$$= 2\pi i \cdot i\pi^2/8 = -\frac{\pi^3}{4}.$$

NOTE: $\int_B h\,dz \longrightarrow J$ AS $\varepsilon \to 0$, $R \to \infty$.

WE COMPUTE: $$\int_D h\,dz = \int_{-R}^{-\varepsilon} \frac{g^2(x)}{1+x^2}\,dx \underset{y=-x}{=} \int_R^{\varepsilon} \frac{g^2(-y)(-dy)}{1+y^2}$$

$$= \int_{\varepsilon}^{R} \frac{(\ln(y)+\pi i)^2}{1+y^2}\,dy$$

$$= \int_{\varepsilon}^{R} \frac{\overset{(1)}{\ln^2(y)} + \overset{(2)}{2\pi i \ln(y)} - \overset{(3)}{\pi^2}}{1+y^2}\,dy$$

$$\rightarrow \overset{(1)}{J} + \overset{(2)}{0} + \overset{(3)}{(-\pi^2)\frac{\pi}{2}}$$

<u>EXERCISE</u> $$\int_{C(\varepsilon)} h\,dz \xrightarrow[\varepsilon\to 0]{} 0$$

$$\int_{C(R)} h\,dz \xrightarrow[R\to\infty]{} 0$$

<u>SO</u>: $$2\pi i \,\mathrm{RES}(h,i) = -\frac{\pi^3}{4} = 2J + \left(-\frac{\pi^3}{2}\right)$$

<u>SO</u>: $\boxed{J = \pi^3/8} \approx 3.8757845\cdots \approx 4$ $\square$

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(2) <u>EXERCISES</u>:

1) $$I = \int_0^{\infty} \frac{1}{\sqrt{1+x^4}}\,dx$$

2) $$I_c = \int_{-\infty}^{\infty} \frac{\cos^2(x)}{1+x^2}\,dx, \quad I_s = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{1+x^2}\,dx$$

3) $I_a = \int_0^\infty \frac{\ln(ax)}{1+x^2}\,dx$, $a > 1$

4) $I_a = \int_0^\infty \frac{x^a}{1+x^2}\,dx$

SKIPPED

5) $I_2 = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$, $I_n = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n}$

6) $I = \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx$ [HARDY'S ARTICLE]

HOW TO DO THESE?? [VIA COMPLEX ANALYSIS, MIND!]

1) $I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{3/2}}\,dx$ } CAN BE DONE VIA $x = \tan(\theta)$, ETC.

2) $J = \int_{-\infty}^{\infty} \frac{\exp(-\sqrt{1+x^2})}{(1+x^2)^2}\,dx$

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③ ENTIRE FUNCTIONS

DEF: SUPPOSE $h : \mathbb{C} \longrightarrow \mathbb{C}$ IS HOLOMORPHIC. CALL h ENTIRE. [DEFINED ON "ENTIRE PLANE"]

LEMMA: SUPPOSE $h : \mathbb{C} \to \mathbb{C}$ ENTIRE. THEN ANY SERIES EXPANSION of h CONVERGES IN ENTIRE PLANE. $\square$

LIOUVILLE'S THM: SUPPOSE h ENTIRE AND $|h|$ BOUNDED. THEN h CONSTANT.

PROOF: FIX $M > 0$ SO THAT $|h(z)| \leq M$ FOR ALL $z \in \mathbb{C}$.

SUPPOSE $h(z) = \sum_i a_n z^n$. SO $a_n = \frac{1}{2\pi i} \int_{C(R)} \frac{h(z)}{z^{n+1}}\,dz$

SO $|a_n| \le M/R^n$ [CAUCHY ESTIMATE]. THIS HOLDS FOR ALL n, R. SO $|a_0| \le M$, $|a_n| = 0$, FOR $n \geq 1$. SO $h \equiv h(0)$ $\square$

(4) THE FUNDAMENTAL THEOREM of ALGEBRA:

SUPPOSE $p \in \mathbb{C}[z]$ IS A POLYNOMIAL. SUPPOSE p NON-CONST. **THEN** p HAS A **ROOT** ($z_0 \in \mathbb{C}$ SO THAT $p(z_0) = 0$).

PROOF: SUPPOSE p HAS NO ROOT. SO $1/p$ IS ENTIRE HOLOMORPHIC. ALSO, IF p HAS DEGREE d THEN $\left|\frac{1}{p(z)}\right| \le K \cdot \frac{1}{|z|^d}$ FOR $|z|$ LARGE. SO $|1/p|$ BOUNDED, SO $1/p$, THUS p, CONSTANT. $\square$

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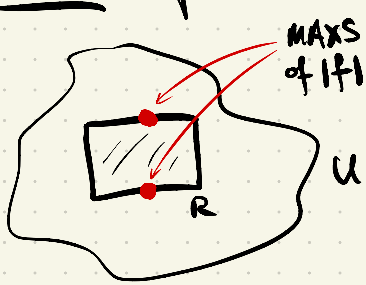
(5) THE MAXIMUM MODULUS PRINCIPLE

THM: SUPPOSE $U \subset \mathbb{C}$ DOMAIN, $R \subset U$ REGION [SO NONEMPTY, COMPACT, NICE BOUNDARY] SUPPOSE $f: U \longrightarrow \mathbb{C}$ HOLOMORPHIC. IF f NON-CONSTANT THEN THE MAX of $|f|$, ON R, IS NOT REALISED IN THE **INTERIOR** of R.

PICTURE



PROOF: NEXT TIME!