2025-11-25 LECTURE 24 SAUL SCHLEIMER MA3B8

① BRANCHES of LOGARITHM:

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LEMMA: SUPPOSE U A DOMAIN. SUPPOSE $H_1(U) \cong 0$. SUPPOSE $f: U \to \mathbb{C}$ HOLOMORPHIC. THEN THERE IS SOME $F: U \to \mathbb{C}$ HOLOMORPHIC SO THAT $F' = f$. [f HAS PRIMITIVES IN U].

SKIP

PROOF: U IS CONNECTED SO PATH-CONNECTED SO CONTOUR-CONNECTED. FIX ANY $z_0 \in U$. FOR ANY $w \in U$ PICK CONTOUR $\gamma: [0,1] \to U$

$\gamma(0) = z_0$, $\gamma(1) = w$

DEFINE $F(w) = \int_\gamma f\, dz$. $F(w)$ INDEP of CHOICE of γ BY CAUCHY AND $H_1(U) \cong 0$.

EXERCISE $F' = f$. □

LOGARITHM THM: SUPPOSE U DOMAIN, $H_1(U) \cong 0$. SUPPOSE $f: U \to \mathbb{C}^\times$ HOLOMORPHIC. THEN HAVE HOLOMORPHIC $g: U \to \mathbb{C}$ SO THAT $f = \mathrm{EXP} \circ g$.

WE CALL g A **BRANCH** of $\mathrm{LOG} \circ f$. AS A SPECIAL CASE: SUPPOSE $U \subset \mathbb{C}^\times$, $H_1(U) \cong 0$. TAKE $f = \mathrm{ID}_U$. THEN g SOLVES $\mathrm{ID}_U = \mathrm{EXP} \circ g$. SO WE WRITE $g = \mathrm{LOG}$ A **BRANCH** of LOG. THIS PROVES (FINALLY!) THAT: $\log(|z|) + i \underbrace{\arg(z)}_{\text{BRANCH of ARG.}}$ IS HOLOMORPHIC.

PROOF of LOGARITHM THM: THE RATIO $f'/f : U \to \mathbb{C}$ IS HOLOMORPHIC IN U. LET $G : U \to \mathbb{C}$ BE A PRIMITIVE FOR f'/f. FIX ANY $z_0 \in U$.

SET $k = \dfrac{\exp(G(z_0))}{f(z_0)}$. PICK ANY $K \in \log(k)$

DEFINE $g = G - K$. SO $g' = G' = f'/f$. NOW:

$$\frac{(\exp \circ g)(z_0)}{f(z_0)} = \frac{\exp(G(z_0) - K)}{f(z_0)}$$

$$= \frac{\exp(G(z_0))}{f(z_0)} \cdot \frac{1}{k} = k \cdot \frac{1}{k} = 1. \checkmark$$

DEFINE: $h(z) = \dfrac{\exp(g(z))}{f(z)}$. SO $h(z_0) = 1$.

$$h'(z) = \frac{f(z) \cdot \exp(g(z)) \cdot g'(z) - f'(z) \cdot \exp(g(z))}{(f(z))^2}$$

$$= \frac{\exp(g(z))}{(f(z))^2} \cdot \left[f(z) \cdot \frac{f'(z)}{f(z)} - f'(z) \right] = 0.$$

SO $h \equiv 1$. SO $f(z) = \exp(g(z))$. $\square$

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② A LOGARITHMIC INTEGRAL:

WE CONSIDER

$$I = \int_0^\infty \frac{\ln(x)}{1+x^2}\, dx.$$

THIS CONVERGES NEAR INFINITY BECAUSE $\ln(x)$ IS DOMINATED BY $x^{1+\varepsilon}$ AND $\int_1^\infty \frac{dx}{x^{1+\varepsilon}}$ CONVERGES.

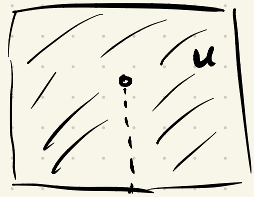
SO SET $G=\int_1^\infty \frac{\ln(x)}{1+x^2}\,dx$. TAKE $u=1/x$ SO $x=1/u$ AND $dx=\frac{-du}{u^2}$.

SO: $G=\int_1^0 \frac{\ln(1/u)}{1+1/u^2}\left(\frac{-du}{u^2}\right)=\int_0^1 \frac{-\ln(u)}{1+u^2}\,du$. } SO I CONVERGES NEAR ZERO.

ALSO: $I=\int_0^\infty \frac{\ln(x)}{1+x^2}\,dx=-G+G=0$.

MORE SUBTLE: $J=\int_0^\infty \frac{\ln^2(x)}{1+x^2}\,dx$ } HERE $\ln^2(x)=(\ln(x))^2$. THIS CONVERGES AS BEFORE. BUT NOW $J>0$.

SET $U=\mathbb{C}(^{3\pi}/_2)$ BE THE CUT PLANE



LET g BE THE BRANCH of LOG IN U SO THAT $g(1)=0$. SO FOR $x>0$ REAL HAVE $g(x)=\ln(x)$

AND $g(-x)=\ln(x)+\pi i$ [$\pi i=g(-1)$ "$=\log(-1)$"]

SO: $h(z)=\frac{g^2(z)}{1+z^2}$ HOLOMORPHIC IN U AND HAS SIMPLE POLE AT $z=i\in U$.

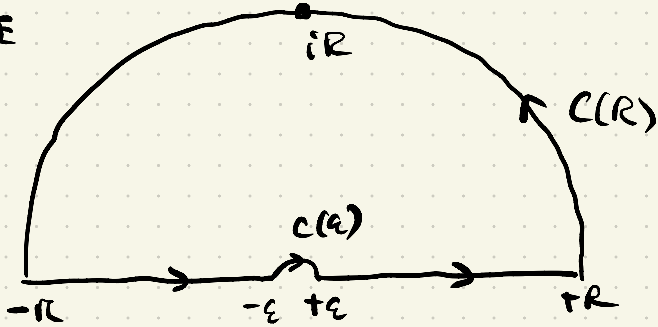
$$\mathrm{RES}(h,i)=\lim_{z\to i}(z-i)\frac{g^2(z)}{(z-i)(z+i)}=\frac{g^2(i)}{2i}=\frac{\left(\frac{\pi i}{2}\right)^2}{2i}=\frac{\pi^2 i}{8}$$

WE NOW GIVE A VERSION of THE KEYHOLE CONTOUR

FIX $0<\varepsilon<1<R$, ε SMALL, R LARGE.

LET $C(\varepsilon)$, $C(R)$ BE THE UPPER HALVES of $C(0;\varepsilon)$, $C(0;R)$. SET $B(\varepsilon,R)=[\varepsilon,R]$, $D(\varepsilon,R)=[-R,-\varepsilon]$

PICTURE



FINISH TOMORROW.