2025-11-14 MA3B8 LECTURE 17 SAUL SCHLEIMER

(1) LAURENT SERIES [BI-SERIES? TWO WAY SERIES? ALL THE POWERS SERIES?]

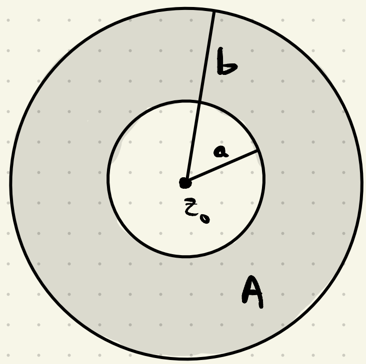
DEF: SUPPOSE $(a_n)_{n\in\mathbb{Z}} \subseteq \mathbb{C}$ IS A BI-INFINITE SEQ. DEFINE THE LAURENT SERIES $\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$ ABOUT z_0 WITH COEFF (a_n).

DEFINE $a, b \in \mathbb{R}\cup\{\infty\}$ BY

$$\frac{1}{b} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \quad \text{AND} \quad a = \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}$$

EXERCISE: SUPPOSE $a < b$. PROVE $\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$ CONVERGES UNIFORMLY ON CPT SUBSETS of $A(z_0;a,b) = A$

PICTURE



THE RESULTING FUNCTION

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$$

IS HOLOMORPHIC ON A. WE OBTAIN $f': A \to \mathbb{C}$ BY TERM-BY-TERM DIFFERENTIATION.

DEFINE a_{-1} THE RESIDUE of THE LAURENT SERIES.

LEMMA: SUPPOSE $a < r < b$. SUPPOSE $C = C(z_0; r)$. THEN:

$$\int_C f\,dz = 2\pi i\, a_{-1}. \qquad \square$$

WE GENERALISE "CAUCHY'S THM" TO OBTAIN "LAURENT'S THM"

<u>THM</u>: SUPPOSE THAT U IS A DOMAIN. SUPPOSE THAT $f: U \to \mathbb{C}$ IS HOLOMORPHIC. SUPPOSE $A = A(z_0; a, b)$ AND $\bar{A} \subset U$. SUPPOSE $a \leq r \leq b$ AND $C = C(z_0, r)$. WE DEFINE

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz \quad \text{FOR } n \in \mathbb{Z}$$

THEN THE LAURENT SERIES $\sum_{n \in \mathbb{Z}}' a_n (z-z_0)^n$ CONVERGES UNIFORMLY ON $\bar{A}$ TO $f|\bar{A}$. □

② POLES

THE REMOVABLE SINGULARITY THM GIVES THE FOLLOWING:

<u>COROLLARY</u>: SUPPOSE $U \subset \mathbb{C}$ A DOMAIN. SUPPOSE $z_0 \in U$. SUPPOSE $f: U^\times = U - \{z_0\} \to \mathbb{C}$ HOLOMORPHIC. SUPPOSE WE HAVE SOME $N \geq 0$ SO THAT $\lim_{z \to z_0} (z-z_0)^{N+1} f(z) = 0$.

<u>THEN</u>: $(z-z_0)^N f(z)$ HAS A HOLO. EXTENSION TO U. □

LET $h: U \to \mathbb{C}$ BE THIS EXTENSION. SO h HAS A SERIES EXPANSION $\sum_{n \geq 0}' a_n (z-z_0)^n$ IN SOME BALL $B = B(z_0; R)$. DIVIDE TO FIND

$$f(z) = \sum_{n=-N}^{\infty}{}' a_{n+N} (z-z_0)^n \quad \text{IN } B^\times.$$

<u>NOTATION</u>: SUPPOSE $f(z) = \sum_{n=N}^{\infty}{}' a_n (z-z_0)^n$ AND $a_N \neq 0$.
$(N \in \mathbb{Z})$

THEN WE WRITE $\mathrm{ORD}(f, z_0) = N$.

SAY: IF $\mathrm{ORD}(f, z_0) > 0$ f HAS A ZERO AT z_0

IF " $= 1$ " SIMPLE ZERO "

IF " $= -1$ " SIMPLE POLE "

IF " < 0 " POLE ".

EXAMPLE: $f(z) = \frac{1}{z^N}$ HAS A POLE OF ORDER N AT $z_0 = 0$.
[NOTE SIGN CHANGE. SORRY!].

EXERCISE: f HAS POLE OF ORDER N AT z_0 IFF

$\lim_{z \to z_0} (z - z_0)^{N+1} f(z) = 0$ AND N IS SMALLEST SUCH POWER.

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③ ESSENTIAL SINGULARITIES:

DEF: SUPPOSE U A DOMAIN, $z_0 \in U$, $f: U^\times \to \mathbb{C}$ HOLOMORPHIC. SUPPOSE THE LAURENT SERIES FOR f ABOUT z_0 HAS INF MANY NEG POWERS. SAY f HAS AN ESSENTIAL SINGULARITY AT z_0.

EXAMPLE: $f(z) = \mathrm{EXP}(1/z) = \sum_{n \le 0} \frac{1}{(-n)!} z^n = \sum_{n \ge 0} \frac{1}{n!} z^{-n}$

NOTE $z^N \cdot \mathrm{EXP}\left(\frac{1}{z}\right) = \dfrac{\mathrm{EXP}(1/z)}{(1/z)^N} = \dfrac{\mathrm{EXP}(w)}{w^N} \longrightarrow \infty$ AS $w \to \infty$ IN $\mathbb{R}$.

SO $\mathrm{EXP}(1/z)$ "HAS A POLE OF INF ORDER" AT $z_0 = 0$.

TRICHOTOMY: SUPPOSE $f: U^\times \longrightarrow \mathbb{C}$ HOLOMORPHIC. THEN z_0 IS EITHER
- REMOVABLE (f EXTENDS)
- A POLE OF FIN ORDER OR
- AN ESSENTIAL SINGULARITY. $\square$.