2025-11-11 LECTURE 16 SAUL SCHLEIMER MA3B8

(1) IDENTITY THEOREM

THM: SUPPOSE U IS A DOMAIN. SUPPOSE $A \subset U$ HAS AN ACCUMULATION POINT IN U. SUPPOSE $f|A = g|A$. THEN $f = g$.

PROOF: NOTE $f-g : U \to \mathbb{C}$ IS HOLOMORPHIC. SO EITHER $f-g \equiv 0$ OR $Z(f-g)$ ISOLATED IN U. BUT $A \subset Z(f-g)$ SO $f-g \equiv 0$ SO $f=g$. □

---

(2) REMOVING REMOVABLE SINGULARITIES

SUPPOSE X IS A SET. SUPPOSE $Y \subset X$. SUPPOSE f IS A FUNCTION ON Y, F IS A FUNCTION ON X. IF $F|Y = f$ THEN WE CALL F AN EXTENSION OF f
f A RESTRICTION OF F.

IF X, Y HAVE STRUCTURE THEN WE MIGHT HAVE "CONTINUOUS EXTENSIONS" OR "HOLOMORPHIC" OR ETC.

EXAMPLE: IF U, V DOMAINS, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ HOLOMORPHIC, $f|U \cap V = g|U \cap V$, AND $h(z) = \begin{cases} f(z) & z \in U \\ g(z) & z \in V \end{cases}$

THEN PATCHING LEMMA TELLS US THAT h IS A HOLOMORPHIC EXTENSION OF f (AND OF g).

THM: [REMOVABLE SINGULARITY] SUPPOSE $U \subset \mathbb{C}$ IS A DOMAIN. SUPPOSE $z_0 \in U$. SUPPOSE $U^\times = U - \{z_0\}$. SUPPOSE $f : U^\times \to \mathbb{C}$ IS HOLOMORPHIC. THE FOLLOWING ARE EQUIVALENT:

(1) f HAS A HOLOMORPHIC EXTENSION TO U

(2) f " " CONTINUOUS " " "

(3) $|f|$ IS BOUNDED IN $B^\times$ FOR $B = B(z_0, \varepsilon)$, FOR SOME $\varepsilon > 0$

(4) $\lim_{z \to z_0} (z - z_0) f(z) = 0$.

NON EXAMPLE $f(z) = 1/\sqrt{z}$ ↑ UP

<u>NON-EXAMPLE</u>: $f : \mathbb{C}^\times \to \mathbb{C}$ DEF BY $f(z) = 1/z$ DOES NOT HAVE A HOLOMORPHIC EXTENSION TO ALL of $\mathbb{C}$.

<u>PROOF of THM</u>: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) ARE EXERCISES.

<u>(4) $\Rightarrow$ (1)</u>: WE DEFINE $h : U \to \mathbb{C}$ BY

$$h(z) = \begin{cases} 0, & \text{if } z = z_0 \\ (z-z_0)^2 f(z), & \text{if } z \neq z_0 \end{cases}$$

NOTE h IS HOLOMORPHIC ON $U^\times$. WE COMPUTE

$$h'(z_0) = \lim_{z \to z_0} \frac{(z-z_0)^2 f(z) - 0}{(z \cancel{- z_0})} = \lim_{z \to z_0} (z - z_0) f(z) = 0.$$

SO $h' : U \to \mathbb{C}$ IS A FUNCTION. SO h IS HOLOMORPHIC.

[NOTE: WE DIDN'T HAVE TO PROVE h' IS CTS!]

SO h IS ANALYTIC. SUPPOSE $h(z) = \sum a_n (z - z_0)^n$ NEAR z_0. SINCE $h(z_0) = h'(z_0) = 0$ HAVE $a_0 = a_1 = 0$. IF $\mathrm{ORD}(h, z_0) = \infty$ THEN $f(z) \equiv 0$ IN $U^\times$ SO EXTEND BY ZERO. SUPPOSE $N = \mathrm{ORD}(h, z_0) < \infty$. NOTE $N \geq 2$.

NOW <u>FACTOR</u>

$$h(z) = (z - z_0)^N \cdot g(z) \begin{cases} \text{WITH } g \text{ HOLOMORPHIC} \\ \text{AND } g(z_0) \neq 0 \end{cases}$$

DEDUCE THAT $(z - z_0)^{N-2} \cdot g(z)$ IS THE DESIRED HOLOMORPHIC EXTENSION of f TO ALL of U. $\square$