1 PATCHING

LEMMA: SUPPOSE U, V ARE DOMAINS. SUPPOSE $U \cap V \neq \emptyset$. THEN $U \cup V$ IS A DOMAIN. □

EXERCISE: SUPPOSE U A DOMAIN. SUPPOSE $z_0 \in U$. THEN $U^\times = U - \{z_0\}$ IS A DOMAIN. □

LEMMA: SUPPOSE U, V ARE DOMAINS. SUPPOSE $U \cap V \neq \emptyset$. SUPPOSE $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ ARE HOLOMORPHIC. SUPPOSE $f|U \cap V = g|U \cap V$. THEN $h : U \cup V \to \mathbb{C}$, DEFINED BY:

$$h(z) = \begin{cases} f(z), & z \in U \\ g(z), & z \in V \end{cases}$$

IS HOLOMORPHIC.

PROOF: HOLOMORPHICITY IS A LOCAL PROPERTY AND $U \cap V$ IS OPEN. □

2 FACTORING:

LEMMA: SUPPOSE $f : U \to \mathbb{C}$ HOLOMORPHIC. SUPPOSE $z_0 \in U$. SUPPOSE $N = \mathrm{ORD}(f, z_0) < \infty$. THEN THERE IS $g : U \to \mathbb{C}$ HOLOMORPHIC SO THAT

(i) $g(z_0) \neq 0$

(ii) $f(z) = (z - z_0)^N \cdot g(z)$ FOR ALL $z \in U$.

PROOF: FIX R SO THAT $B = B(z_0; R) \subset U$. LET $A = (a_n)$ BE THE COEFFICIENTS OF SERIES EXPANSION OF $f|B$.