2025-11-05 LECTURE 14 SAUL SCHLEIMER MA358

① COROLLARIES ① a_n INDEP of CHOICE of $R \in (0, R_0)$.

② $f^{(n)}(z_0) = n! \cdot a_n$

③ $\frac{1}{R_0} \geq \limsup_{n \to \infty} \sqrt[n]{|a_n|}$ } CORRECTED FROM LECTURE.

SO: IF $U = \mathbb{C}$ THEN $R_0 = \infty$ AND $\sum_i a_n (z - z_0)^n$ CONVEGES IN ALL of $\mathbb{C}$.

② THE CONVERSE TO EVERYTHING [MORERA'S THM]

COROLLARY: SUPPOSE $f : U \to \mathbb{C}$ CTS AND INTEGRATES TO ZERO ABOUT TRIANGLES. THEN f IS HOLOMORPHIC.

PROOF: INTEGRATES TO ZERO ABOUT TRIANGLES IMPLIES HAS PRIMITIVES IN DISCS. SUPPOSE F DEFINED IN $B = B(z_0, R)$ AND $F' = f$. SO F IS HOLOMORPHIC. SO F IS ANALYTIC. SO $F' = f$ IS ANALYTIC. SO f IS HOLOMORPHIC. □

③ THE FUNDAMENTAL THM of COMPLEX ANALYSIS

[FOLLOWING BERS]

THEOREM: SUPPOSE THAT $U \subset \mathbb{C}$ IS A DOMAIN. SUPPOSE THAT $f : U \to \mathbb{C}$ IS CONTINUOUS. THEN THE FOLLOWING ARE EQUIVALENT.

① f IS HOLOMORPHIC.

② f INTEGRATES TO ZERO ABOUT TRIANGLES.

③ f HAS PRIMITIVES IN DISCS

④ f INTEGRATES TO ZERO ABOUT CLOSED CONTOURS IN DISCS

⑤ " " " " " ONE-BOUNDARIES

⑥ " " " " " REGIONS

(7) FOR ALL $z_0 \in U$, $R < R_0$, $w \in B = B(z_0, R)$

$$f(w) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{z-w}\, dz$$

(8) f IS ANALYTIC AND, IN $B = B(z_0, R)$ [AS ABOVE]

$$f(z) = \sum_1' a_n (z-z_0)^n \quad \text{FOR} \quad a_n = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

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(4) <u>ZEROS</u> SUPPOSE $f : U \to \mathbb{C}$ HOLOMORPHIC. $z_0 \in U$ IS A <u>ZERO</u> of f IF $f(z_0) = 0$.

<u>DEF:</u> SAY f <u>VANISHES IDENTICALLY</u> IN U IF $f(z) = 0$ FOR ALL z IN U. IF NOT, SAY f IS NOT <u>IDENTICALLY ZERO</u> IN U.

<u>DEF:</u> SUPPOSE $f : U \to \mathbb{C}$ HOLOMORPHIC. SUPPOSE $z_0 \in U$. SUPPOSE $f(z) = \sum_1' a_n (z-z_0)^n$ IS THE SERIES EXPANSION of f AT z_0.

IF ALL $a_n = 0$ SAY f VANISHES TO <u>INFINITE ORDER</u> AT z_0

IF NOT, SET $N = \text{MIN}\{ n \in \mathbb{N} \mid a_n \neq 0 \}$ AND SAY f <u>VANISHES TO ORDER</u> N AT z_0. WE ALSO WRITE $\text{ORD}(f, z_0) = N$ FOR THE "ORDER of VANISHING".

NOTE z_0 IS A ZERO of f IFF $N > 0$.

(5) VANISHING:

LEMMA: SUPPOSE f VANISHES TO INF. ORDER AT z_0.

THEN f VANISHES IDENTICALLY IN U.

PROOF: SUPPOSE $z_1 \in U$. FIX A PATH $\gamma : [0,1] \to U$ WITH $\gamma(0) = z_0$, $\gamma(1) = z_1$.

EXERCISE: FIND POINTS $z_0 = w_0, w_1, \cdots w_k, \cdots w_N = z_1$ IN THE IMAGE of γ SO THAT FOR ALL k

(i) $|w_{k+1} - w_k| < \varepsilon$

(ii) $B(w_k, \varepsilon) \subset U$.

LET $A_k = (a_{k,n})_{n \in \mathbb{N}}$ BE THE COEFFICIENTS of THE SERIES EXPANSION of f AT w_k. SO $A_0 \equiv 0$. SUPPOSE $A_k \equiv 0$. THUS f VANISHES IDENTICALLY IN $B(w_k, \varepsilon)$. SO f VANISHES IDENTICALLY IN A NEIGHBOURHOOD of w_{k+1}. SO $f^{(n)}(w_{k+1}) = 0$ FOR ALL n. SO $A_{k+1} \equiv 0$. BY INDUCTION $A_N \equiv 0$. THUS $f(w_N) = f(z_1) = 0$. $\square$

PICTURE