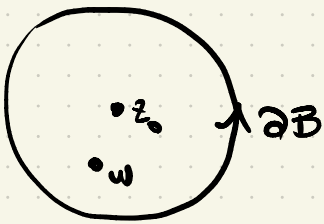
2025-11-04 MA3B8 LECTURE 13 SAUL SCHLEIMER ~55

① THE REPRODUCTION FORMULA

THEOREM: SUPPOSE $f: U \to \mathbb{C}$ HOLOMORPHIC. SUPPOSE $B = B(z_0; R)$ HAS $\overline{B} \subset U$. THEN, FOR ALL $w \in B$, WE HAVE

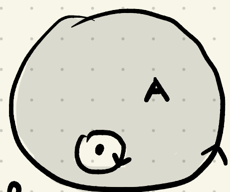
$$f(w) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{z - w}\, dz.$$

PICTURE:



IF WE USE THE WRONG ORIENTATION WE GET A SIGN ERROR.

PROOF: FIX $\varepsilon > 0$, SMALL ENOUGH SO THAT, FOR $B' = B(w; \varepsilon)$ WE HAVE $\overline{B'} \subset B$. LET A BE THE REGION $A = \overline{B} - B'$. SO $\partial A = \partial B - \partial B'$.



NOTE $g(z) = f(z)/_{z-w}$ IS HOLOMORP. IN $U - \{w\}$.

SO $$0 = \int_{\partial A} g\, dz = \int_{\partial A} \frac{f(z)}{z-w}\, dz = \int_{\partial B - \partial B'} \frac{f(z)}{z-w}\, dz$$

THUS: $$\int_{\partial B} \frac{f(z)}{z-w}\, dz = \int_{\partial B'} \frac{f(z)}{z-w}\, dz.$$ NOW DEFINE $\gamma: [0, 2\pi] \to U$ BY $\gamma(\theta) = w + \varepsilon \cdot e^{i\theta}$.

WE COMPUTE: $$\left| \frac{1}{2\pi i} \int_{\partial B'} \frac{f(z)}{z-w}\, dz - f(w) \right| =$$

$$= \left| \frac{1}{2\pi i} \int_{\partial B'} \frac{f(z) - f(w)}{z - w}\, dz \right| = \frac{1}{2\pi} \left| \int_{\partial B'} \frac{(f'(w) + \rho(z))(z-w)}{z-w}\, dz \right|$$

$$\leq \frac{1}{2\pi} M \cdot \underbrace{\varepsilon \cdot 2\pi}_{L(\gamma)} \quad \text{FOR } M = \max\left\{ |f'(w) + \rho(z)| : z \in \overline{B} \right\}$$

$$= M \cdot \varepsilon.$$ SINCE M FIXED AND $\varepsilon > 0$ IS ARBITR.

DEDUCE $f(w) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{z-w}\, dz$ $\square$.

RESTATEMENT: WE CAN RECOVER f INSIDE of B FROM ITS BOUNDARY VALUES $f|\partial B$

(2) THE MEAN VALUE THEOREM

COROLLARY: $f : U \to \mathbb{C}$ HOLOMORPHIC, $B = B(z_0, R)$ WITH $\overline{B} \subset U$. THEN

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta.$$

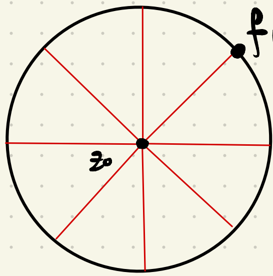
SO: ON A CENTRED DISC, $f(z_0)$ IS THE AVERAGE of ITS BOUNDARY VALUES.

PROOF: $f(z_0) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{z - z_0}\, dz$

$$= \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{\cancel{Re^{i\theta}}} \cancel{i} \cancel{Re^{i\theta}}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta \qquad \square$$

PICTURE



$f(z_0+Re^{i\theta})$

$f(z_0)$ IS THE AVERAGE of ITS BOUNDARY VALUES ON ANY DISC.

THIS IS VERY FALSE FOR FUNCTIONS $g:\mathbb{R}^2\to\mathbb{R}$ EG $g(x,y)=x^2+y^2$.

RMK: WE WILL NOT COVER HARMONIC FUNCTIONS.

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③ ANALYTIC FUNCTIONS.

SUPPOSE $f:U\to\mathbb{C}$ A FUNCTION. WE SAY f IS **ANALYTIC** IF FOR ALL $z_0\in U$ THERE IS SOME $R>0$ AND $(a_n)_{n=0}^{\infty}\subset\mathbb{C}$ SO THAT

(1) $B=B(z_0;R)\subset U$ AND

(2) $f(z)=\sum_n a_n(z-z_0)^n$ FOR ALL $z\in B(z_0;R)$

THUS $R\geq$ RADIUS of CONVERGENCE of $(a_n)_{n=0}^{\infty}$.

LEMMA: ANALYTIC $\Rightarrow$ HOLOMORPHIC.

PROOF: $f'(z)=\sum_n na_n(z-z_0)^{n-1}$ IN B $\square$

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④ THE CONVERSE

THEOREM: HOLOMORPHIC $\Rightarrow$ ANALYTIC.

PROOF: SUPPOSE $f:U\to\mathbb{C}$ IS HOLOMORPHIC. FIX $z_0\in U$. DEFINE $R_0=\sup\{R\in\mathbb{R}\mid B(z_0,R)\subset U\}$ [$R_0=\infty$ IFF $U=\mathbb{C}$] FIX $R<R_0$, $R>0$. DEFINE $B=B(z_0,R)$. SO $\bar{B}\subset U$.

DEFINE $a_n = \frac{1}{2\pi i}\int_{\partial B} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ FOR $n = 0, 1, 2, \ldots$

DEFINE $M = \max\{|f(z)| : z \in B\}$.

So: $|a_n| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(z_0+Re^{i\theta})|}{R^{n+1}} \cdot R\,d\theta \le \frac{M}{R^n}$. ["CAUCHY ESTIMATE"]

THUS: $F(z) = \sum_0^\infty a_n (z-z_0)^n$ CONVERGES (UNIF ON COMPACT SETS) IN $B = B(z_0; R)$.

CLAIM: $f(z) = F(z)$ IN B.

PROOF:

$$F(w) = \sum_0^\infty a_n (w-z_0)^n = \sum_0^\infty \int_{\partial B} \frac{f(z)}{(z-z_0)^{n+1}} \cdot (w-z_0)^n\,dz$$

$$= \int_{\partial B} \sum^\infty \frac{f(z)}{(z-z_0)}\left(\frac{w-z_0}{z-z_0}\right)^n dz$$

$$= \int_{\partial B} \frac{f(z)}{(z-z_0)}\left[\sum^\infty \left(\frac{w-z_0}{z-z_0}\right)^n\right] dz$$

$$= \int_{\partial B} \frac{f(z)}{(z-z_0)} \frac{1}{1-\left(\frac{w-z_0}{z-z_0}\right)}\,dz$$

$$= \int_{\partial B} \frac{f(z)}{\cancel{(z-z_0)}} \frac{\cancel{(z-z_0)}}{z-\cancel{z_0}-(w-\cancel{z_0})}\,dz = \int_{\partial B} \frac{f(z)}{z-w}\,dz = f(w). \qquad \square$$