2025-10-29 LECTURE 11 MA 3B8 SAUL SCHLEIMER

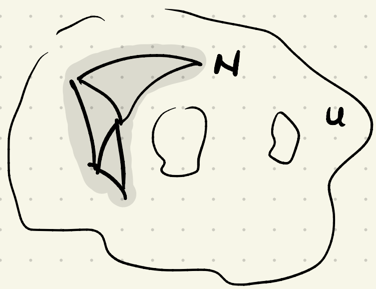
(1) A VERSION OF CAUCHY'S THEOREM

THEOREM: SUPPOSE U IS A DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ IS CONTINUOUS AND INTEGRATES TO ZERO ABOUT CONTOURS IN DISCS. SUPPOSE $\Gamma \in B_1(U)$ IS PIECEWISE C^1. THEN
$$\int_\Gamma f\, dz = 0.$$

COROLLARY: U A DOMAIN, f HOLOMORPHIC, $\Gamma \in B_1(U)$ PIECEWISE C^1. THEN $\int_\Gamma f\, dz = 0$.

"f INTEGRATES TO ZERO ABOUT "BOUNDARIES" "

PROOF: SUPPOSE $d = \sum_1 a_k \sigma_k^2 \in C_2(U)$ HAS $\partial d = \Gamma$. NOTE $D = \bigcup_k \text{IMAGE}(\sigma_k^2)$ IS COMPACT AND IN U. SO HAVE $\varepsilon > 0$ SO THAT $N(D, \varepsilon) = \varepsilon$-NEIGH LIES IN U. PICTURE:



BY LEMMA HAVE $N >> 0$ SO THAT ALL SIMPLICES IN $\text{SUB}_2^N(d)$ HAVE IMAGE WITH DIAMETER LESS THAN ε.

OTHER LEMMA GIVES $\text{SUB}_1^N(\Gamma) = \text{SUB}_1^N(\partial d)$
$$= \partial\, \text{SUB}_2^N(d) \in B_1(U).$$

BY LINEARITY IN DOMAIN $$\int_\Gamma f\, dz = \int_{\text{SUB}_1^N(\Gamma)} f\, dz.$$

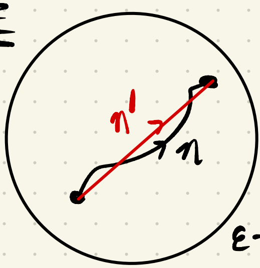
NOW WE CAN STRAIGHTEN: SUPPOSE $SUB_2^N(d) = \sum_1^{!} \sigma_k^2$.
SO $DIAM(IMAGE(\sigma_k^2)) < \varepsilon$. SO $IMAGE(\sigma_k^2) \subset B(\sigma_k(v_0), \varepsilon)$ IS CONTAINED IN U. SET $\tau_k = STR(\sigma_k)$. SO τ_k HAS IMAGE IN $B(\sigma_k(v_0), \varepsilon)$. SO $IMAGE(\tau_k) \subset U$. SET $d' = \sum_1^{!} \tau_k = STR(SUB^N(d))$. SO $d' \in C_2(U)$.
SET $\Gamma' = \partial d'$. SO $\Gamma' = \sum_1^{!} \partial \tau_k$.

BY HYPOTHESIS $$\int_{\Gamma'} f\, dz = \sum_1^{!} \int_{\partial \tau_k} f\, dz = 0.$$

SO SUFFICES TO PROVE $$\int_{\Gamma'} f\, dz = \int_{SUB_1^N(\Gamma)}.$$

FOR ANY EDGE η of $SUB_1^N(\Gamma)$ HAVE $\eta' = STR_1(\eta)$ IN Γ'
NOTE DIAMETER of IMAGE of η IS LESS THAN ε.
SO SAME HOLDS FOR η'.

PICTURE



ε-DISC

SO $\eta - \eta'$ IS CLOSED CONTOUR IN A DISC. SO $\int_{\eta-\eta'} f\, dz = 0$.

THUS: $$\int_{\Gamma} f\, dz = \int_{SUB_1^N(\Gamma)} f\, dz = \int_{\Gamma'} f\, dz = 0. \qquad \square$$

EXAMPLE: NOTE $f(z) = 1/z$ IS HOLOMORPHIC IN $\mathbb{C}^\times$. NOTE $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}^\times$, $\gamma(\theta) = e^{i\theta}$ IS A CLOSED CONTOUR $[\gamma \in Z_1(\mathbb{C}^\times)]$ NOTE $\int_\gamma f\, dz = 2\pi i \neq 0$.

SO: γ IS NOT A BOUNDARY. SO $H_1(\mathbb{C}^\times) \neq 0$. $\square$

IN FACT: $H_1(\mathbb{C}^\times) \cong \mathbb{Z}$, GEN BY $[\gamma]$.

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② CAUCHY'S THY, SO FAR

COROLLARY: SUPPOSE THAT U IS A DOMAIN.
SUPPOSE THAT $f : U \longrightarrow \mathbb{C}$ IS CONTINUOUS.

THEN EACH of THE FOLLOWING IMPLIES THE NEXT:

(1) f IS HOLOMORPHIC

(2) f INTEGRATES TO ZERO ABOUT TRIANGLES

(3) f HAS PRIMITIVES IN DISCS IN U.

(4) f INTEGRATES TO ZERO ABOUT CLOSED CONTOURS IN DISCS IN U.

(5) f INTEGRATES TO ZERO ABOUT NULL HOMOLOGOUS CONTOURS IN U.

RMK: $\left.\begin{array}{l}(5) \Rightarrow (4) \\ (4) \Rightarrow (3) \\ (3) \Rightarrow (2)\end{array}\right\}$ ARE EASY (OR AT LEAST, NOT HARD!)

RMK: $(2) \Rightarrow (1)$ IS DIFFICULT! WE'LL RETURN TO THIS.