(1) FROM LAST TIME: $C_n(X) = \{n\text{-CHAINS IN } X\}$

~~ASSESSED WORK~~

~~MS01 LIGHTS~~

$\partial_n : C_n(X) \to C_{n-1}(X)$ BOUNDARY OPERATOR

$Z_n(X) = \mathrm{KER}(\partial_n) = \{n\text{-CYCLES IN } X\}$

$B_n(X) = \mathrm{IMAGE}(\partial_{n+1}) = \{n\text{-BOUNDARIES IN } X\}$

$H_n(X) = Z_n(X) / B_n(X) = \{n\text{-HOMOLOGY CLASSES IN } X\}$.

EXAMPLE: $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$ $\quad H_1(\mathbb{D}) = 0$.

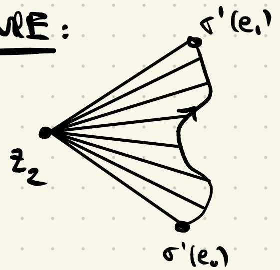
LEMMA: SUPPOSE $U \subset \mathbb{C}$ IS CONVEX. THEN $H_1(U) = 0$.

PROOF: FIX $z_2 \in U$. SUPPOSE $\sigma' : \Delta' \to U$ IS A SINGULAR ONE-SIMPLEX. DEFINE $\mathrm{CON}(\sigma') : \Delta^2 \to U$ BY

$$\mathrm{CON}(\sigma')(\lambda_0, \lambda_1, \lambda_2) = \begin{cases} (1-\lambda_2)\cdot z_2 + \lambda_2 \cdot \sigma'\left(\frac{1}{1-\lambda_2}(\lambda_0, \lambda_1)\right), & \text{if } \lambda_2 \neq 1 \\ z_2, & \text{if } \lambda_2 = 1 \end{cases}$$

WAS INCORRECT IN THE LECTURE

PICTURE:

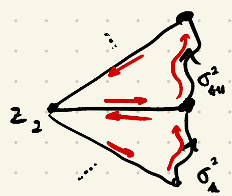


$$\left.\begin{aligned} \mathrm{CON}(\sigma')(e_0) &= \sigma'(e_0) \\ \mathrm{CON}(\sigma')(e_1) &= \sigma'(e_1) \\ \mathrm{CON}(\sigma')(e_2) &= z_2. \end{aligned}\right\}$$

DEFINE $\mathrm{CON} : C_1(U) \to C_2(U)$ BY $\mathrm{CON}(\sum a_k \sigma_k') = \sum a_k \mathrm{CON}(\sigma_k')$.

CLAIM: IF $z \in Z_1(U)$ THEN $\partial(\mathrm{CON}(z)) = z$

PROOF of CLAIM

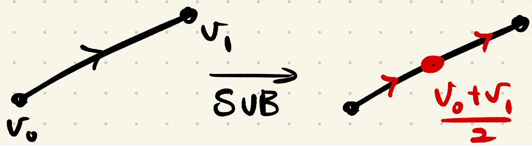


ALL INTERNAL EDGES CANCEL IN PAIRS. $\square$

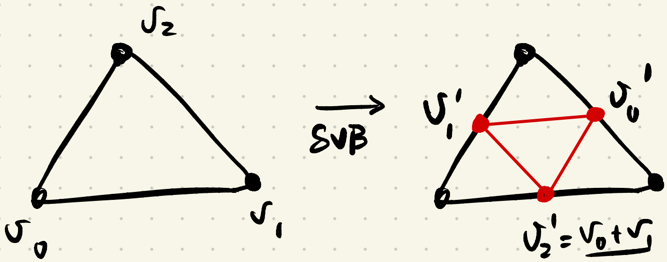
THUS: $Z_1(U) = B_1(U)$ AND SO $H_1(U) = 0$ $\square$

② SUBDIVISION: [BARYCENTRIC SUBDIVISION IS BETTER]

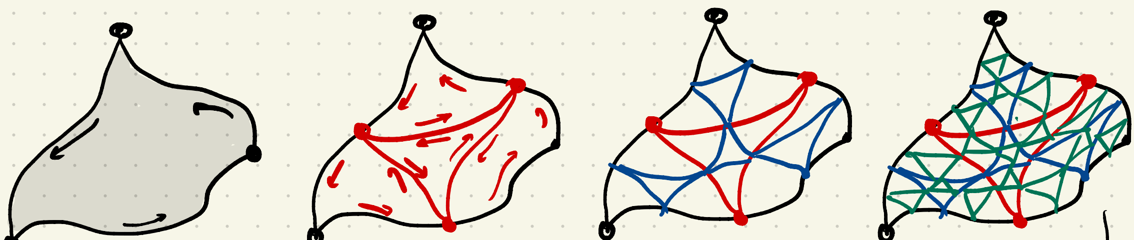
WE SUBDIVIDE Δ^1 OR Δ^2 USING THE MIDPT SUBDIVISION



AND



SUPPOSE σ IS A SINGULAR ONE- OR TWO-SIMPLEX. THEN WE PRECOMPOSE WITH SUBDIVISION TO GET THE CHAIN $SUB(\sigma)$. WE CAN ITERATE SUBDIVISION TO GET



σ $\quad SUB_2(\sigma)$ $\quad SUB_2^2(\sigma)$ $\quad SUB_2^3(\sigma)$.

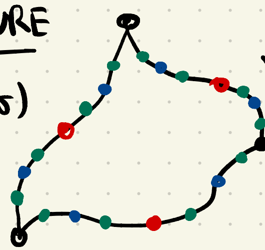
WE DEFINE
$$\left\{ \begin{array}{l} SUB_2 : C_2(U) \longrightarrow C_2(U) \\ SUB_1 : C_1(U) \longrightarrow C_1(U) \end{array} \right\}$$
AND

EXTEND LINEARLY: $SUB_n\left(\sum_1 a_k \sigma_k^n\right) = \sum_1 a_k \, SUB_n(\sigma_k^n)$.

∂_1

PICTURE

$SUB_1^3(\partial\sigma)$



LEMMA: FOR $c \in C_2(U)$ WE HAVE: $\partial SUB_2^N(c) = SUB_1^N(\partial c)$

$\square$

LEMMA: FIX $c \in C_2(U)$. FIX $\varepsilon > 0$. THEN THERE IS $N > 0$ SO THAT, FOR ALL σ_k IN $SUB^N(c)$, WE HAVE

$$\text{DIAMETER}(\text{IMAGE}(\sigma_k)) < \varepsilon. \qquad \square$$

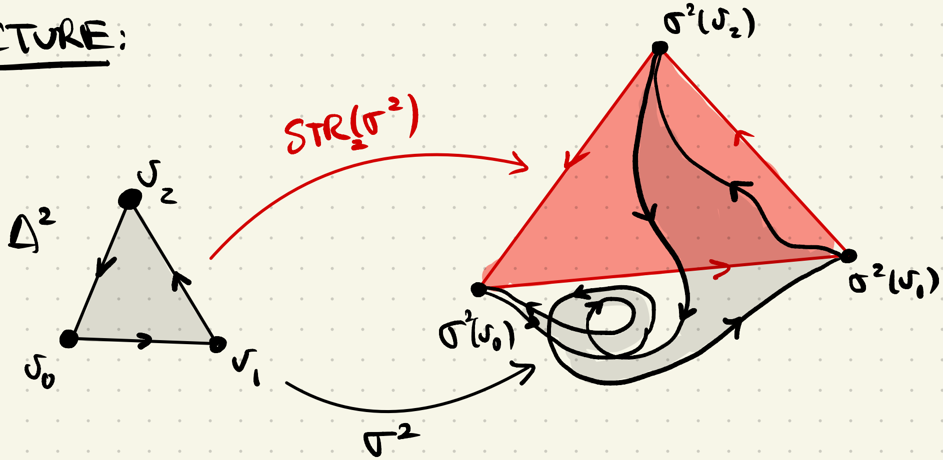
③ STRAIGHTENING : SUPPOSE $U \subset \mathbb{C}$ IS CONVEX DOMAIN.

SUPPOSE $\sigma^n : \Delta^n \longrightarrow U$ IS A SINGULAR n-SIMPLEX.

DEFINE: $STR_n(\sigma^n) : \Delta^n \longrightarrow U$ BY

$$STR_n(\sigma^n)(x_0, x_1, \cdots x_n) = \sum x_k \sigma^n(v_k)$$

PICTURE:



EXERCISE:

AGAIN, EXTEND TO DEFINE $STR_n : C_n(U) \longrightarrow C_n(U)$

RMK: $STR_0 : C_0(U) \longrightarrow C_0(U)$ IS THE IDENTITY!

LEMMA: $STR(\partial c) = \partial(STR(c))$ $\square$

PICTURE :