2025-10-24 LECTURE 9 MA3B8 SAUL SCHLEIMER [PLT]

① SIMILAR TO LAST TIME

PROPOSITION: SUPPOSE $f : U \to \mathbb{C}$ CONTINUOUS. THEN:

① f HAS PRIMITIVES IN U ⊛

⊛ NOT JUST IN DISKS.

IFF

② $\int_\gamma f\, dz = 0$ FOR ALL CLOSED CONTOURS $\gamma : [a,b] \to U$.

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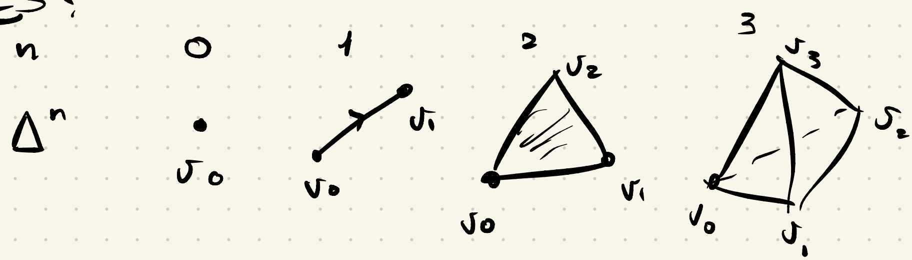
② SINGULAR SIMPLICIES

DEFINE $\Delta^n = \{ (x_k) \in \mathbb{R}^{n+1} \mid x_k \geq 0,\ \sum_i x_k = 1 \}$.

SET $v_k = (0, 0, \ldots, 0, \underset{\uparrow\, k^{th}}{1}, 0, \ldots 0)$

PICTURES:



DEF: SUPPOSE X IS A TOP. SPACE. CALL CONTINUOUS $\sigma^n : \Delta^n \longrightarrow X$ A SINGULAR n-SIMPLEX IN X

DEF: $C_n(X)$ = FREE ABELIAN GROUP GEN BY SINGULAR n-SIMPLICES.

$$= \Big\{ \sum_{k=0}^{N_1} a_k \sigma_k^n \;\Big|\; a_k \in \mathbb{Z},\ \sigma_k^n : \Delta^n \longrightarrow X \text{ CTS} \Big\}.$$

↖ AN n-CHAIN.

PICTURE

1-CHAIN IN $\mathbb{C}$

2-CHAIN IN $\mathbb{C}$

RMK: A CHAIN IS SMOOTH, OR LINEAR, OR PIECEWISE C^1 IF ITS SIMPLICES ARE.

SO: CONTOURS ARE "PIECEWISE C^1 ONE-CHAINS".

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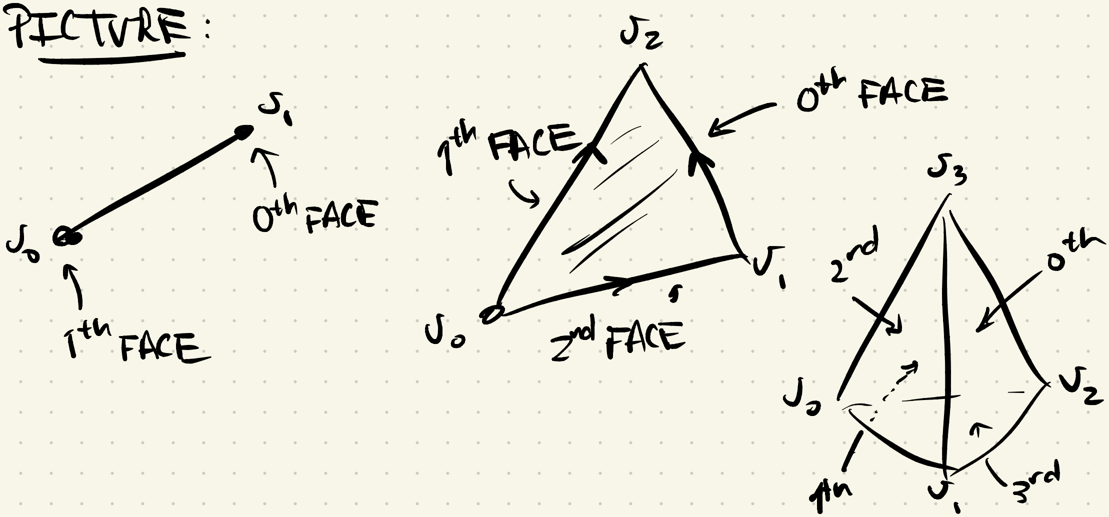
③ FACES: SUPPOSE Δ^n IS THE STD n-SIMPLEX. $v_0, \dots, v_n$ ARE ITS VERTICES. SUPPOSE Δ^{n-1} IS STD $n-1$ SIMPLEX, WITH VERTICES $w_0, \dots, w_{n-1}$.

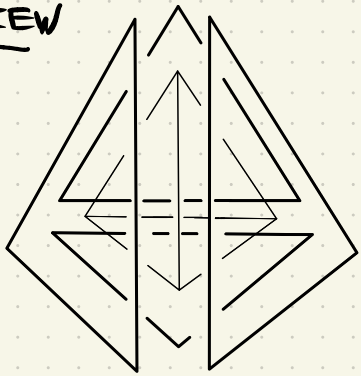
FIX k, $0 \le k \le n$. DEFINE THE k^{th} FACE of Δ^n TO BE THE AFFINE MAP

$$\begin{aligned} \Delta^{n-1} &\longrightarrow \Delta^n \\ \textstyle\sum_i' \lambda_i w_i &\longmapsto \sum_{i<k} \lambda_i v_i + \sum_{i \ge k}' \lambda_i v_{i+1} \end{aligned}$$

PICTURE:



EXPLODED VIEW



WE USE $\sigma^n | [v_0 \cdots \hat{v}_k \cdots v_n]$ TO DENOTE σ^n RESTRICTED TO THE k^{th} FACE (AND REPARAMETRISED BY Δ^{n-1}).

DEFINE:

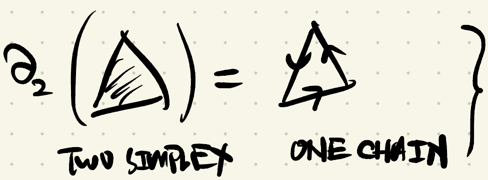
OMITTED VERTEX.

$$\partial_n \sigma^n = \sum_{k=0}^{n} (-1)^k \sigma^n | [v_0 \cdots \hat{v}_k \cdots v_n]$$

DEFINE: $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ TO BE THE LINEAR EXTENSION of ∂_n. SO: $\partial_n c = \partial_n\left(\sum_i a_i \sigma_i^n\right)$

$$= \sum_i a_i \partial_n \sigma_i^n.$$

THIS IS THE n^{th} BOUNDARY OPERATOR



EXERCISE: $\partial_{n-1}\partial_n = 0$.

CLOSED!

EXAMPLE: $\partial_1 \circ \partial_2$ () $= \partial_1$ () $= 0$.

DEF: $Z_n(X) = \{ c \in C_n(X) \mid \partial_n c = 0 \} = \mathrm{KER}(\partial_n)$

$B_n(X) = \{ c \in C_n(X) \mid \exists d \in C_{n+1}(X), \partial_{n+1} d = c \} = \mathrm{IMAG}(\partial_{n+1})$

$z \in Z_n(X)$ IS AN n-<u>CYCLE</u>

$b \in B_n(X)$ IS AN n-<u>BOUNDARY</u>.

<u>LEMMA</u>: $B_n(X) \leq Z_n(X)$. <u>PROOF</u>: $\partial_n \circ \partial_{n+1} = 0$. $\square$.

<u>DEF</u>: $H_n(X) = Z_n(X) / B_n(X)$ "CYCLES MODULO BOUNDARIES".

CALL $[z] \in H_n(X)$ A <u>HOMOLOGY CLASS</u>.

IF $[z] = 0$ CALL z A <u>NULL-HOMOLOGOUS</u> n-CYCLE.

IF $[z] = [z']$ CALL z, z' <u>HOMOLOGOUS</u>. [EQUIV: $[z - z'] = 0$]

<u>DEF</u>: SUPPOSE $U \subset \mathbb{C}$ IS A DOMAIN. SAY U HAS <u>TRIVIAL FIRST HOMOLOGY</u> IF $H_1(U) = 0$.

[ALL CYCLES ARE BOUNDARIES]