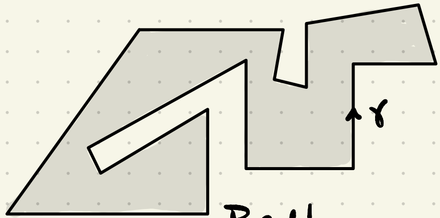
2025-10-22 LECTURE 8 MA3B8 SAUL SCHLEIMER

EXERCISE HOLOMORPHIC FUNCTIONS INTEGRATE TO ZERO ABOUT POLYGONS

[REGION HOMEO TO $\overline{\mathbb{D}}$ WITH PIECEWISE LINEAR BOUNDARY.]



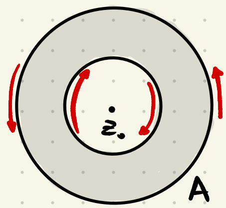
$P \subset U$

① TOO CLEVER BY HALF

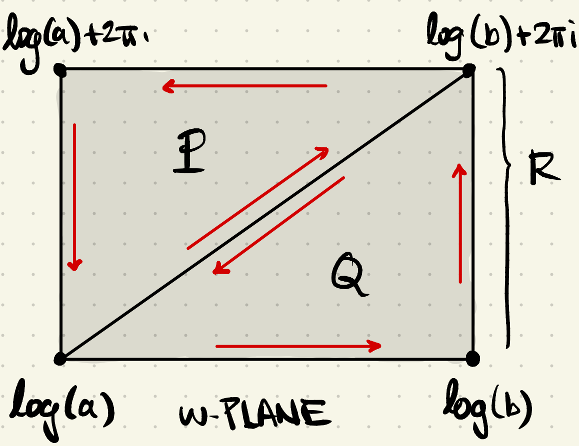
LEMMA: SUPPOSE $f: U \to \mathbb{C}$ IS HOLOMORPHIC. FIX $0 < a < b$. SUPPOSE $A = A(z_0; a, b)$ (ANNULUS) HAS $\overline{A} \subset U$.

THEN $\displaystyle\int_{\partial A} f \, dz = 0$

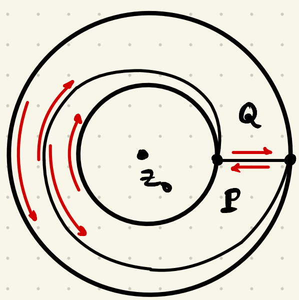
PICTURE



PROOF: DEFINE THE RECTANGLE R BY:



NOTE THAT $\mathrm{EXP} + z_0$ MAPS R TO A



Z-PLANE

SO TAKE $z = \mathrm{EXP}(w) + z_0$

$dz = \mathrm{EXP}(w)\, dw$

AND SO

$$\int_{\partial A} f(z)\,dz = \int_{\partial R} \underbrace{f(\mathrm{EXP}(w)+z_0)\,\mathrm{EXP}(w)}^{\text{HOLOMORPHIC}}\,dw = 0. \qquad \square$$

THAT IS, WE CAN PARAMETRISE ∂A IN A CLEVER (NOT VERY GENERAL) WAY.

QUESTION: WHAT IF $a=0$? LATER

QUESTION: $\int_{\partial B} f\,dz = 0$? YES.

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② **PRIMITIVES**: SUPPOSE U IS A DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ IS A FUNCTION. SUPPOSE $F: U \to \mathbb{C}$ IS HOLOMORPHIC AND $F' = f$. THEN WE SAY "F IS A **PRIMITIVE** FOR f" AND "f HAS A **PRIMITIVE**"

NOTE: THESE ARE JUST ANTIDERIVATIVES

NOTE: IF F, G ARE PRIMITIVES FOR f THEN $F - G$ IS CONSTANT.

[**PROOF**: $(F-G)' = F' - G' = f - f = 0$. AND MVT]

LEMMA: SUPPOSE $B = B(w_0, r)$ IS A DISK. SUPPOSE $f: B \to \mathbb{C}$ IS CONTINUOUS. SUPPOSE f INTEGRATES TO ZERO ABOUT TRIANGLES IN B. THEN f HAS A PRIMITIVE IN B.

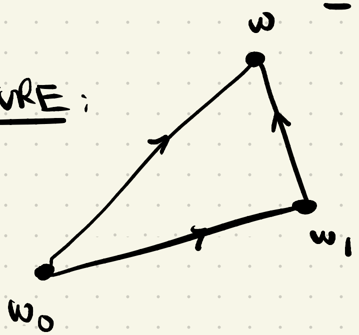
PROOF: FOR $w \in B$, DEFINE $F(w) = \int_{[w_0, w]} f\,dz$.

FIX ANY $w_1 \in B$. WE MUST SHOW $F'(w_1)$ EXISTS AND EQUALS $f(w_1)$.

NOTE: $F(w) - F(w_1) = \int_{[w_0,w]} f\,dz - \int_{[w_0,w_1]} f\,dz$

$$= \int_{[w_1, w]} f\, dz.$$

PICTURE:



WE SUBTRACT $f(w_1)(w-w_1)$ FROM BOTH SIDES TO GET:

$$F(w) - F(w_1) - f(w_1)(w-w_1) = \int_{[w_1,w]} \big(f(z) - f(w_1)\big)\, dz$$

RECALL THAT f IS CONTINUOUS. FIX $\varepsilon > 0$. SO HAVE $\delta > 0$ WITH $|z - w_1| < \delta$ IMPLYING $|f(z) - f(w_1)| < \varepsilon$.

SO $\big|F(w) - F(w_1) - f(w_1)(w-w_1)\big| < \varepsilon \cdot |w - w_1|$. [ML]

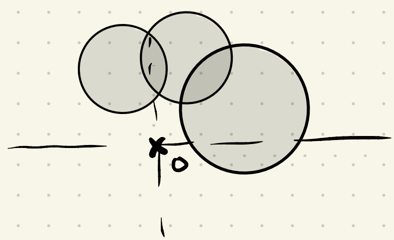
SO $\left| \dfrac{F(w) - F(w_1)}{w - w_1} - f(w_1) \right| < \varepsilon$. $\square$

COROLLARY: SUPPOSE $f : U \to \mathbb{C}$ IS HOLOMORPHIC. SUPPOSE $B \subset U$ IS A DISK. THEN f HAS A PRIMITIVE IN B. $\square$.

CAREFUL: IN B, NOT IN U!

EXAMPLE: $f : \mathbb{C}^\times \to \mathbb{C}$ DEFINED BY $f(z) = \frac{1}{z}$.

THEN f HAS PRIMITIVES IN DISKS $B \subset \mathbb{C}^*$ BUT NOT IN $\mathbb{C}^\times$. PICTURE:



DISCUSSED "ANALYTIC CONTINUATION".

QUESTION ASKED EARLIER [ADDED TO NOTES AFTER LECTURE]

LEMMA: SUPPOSE $f : U \to \mathbb{C}$ HOLOMORPHIC. SUPPOSE $B = B(z_0; R)$ AND $\overline{B} \subset U$. THEN $\int_{\partial B} f \, dz = 0$.

PROOF 1: DEFINE $A_r = A(z_0; r, R)$. SO $\overline{A_r} \subset U$ AND $\int_{\partial A_r} f \, dz = 0$. NOW APPLY ML-INEQUALITY AND TAKE $r \to 0$. $\square$

PROOF 2: FIX $R' > R$ SO THAT $B' = B(z_0; R') \subset U$. SO $\overline{B} \subset B'$. FIX F A PRIMITIVE FOR f IN B'. SET

$$\gamma : [0, 2\pi] \longrightarrow U$$
$$\theta \longmapsto Re^{i\theta} + z_0.$$

COMPUTE:

$$\int_{\partial B} f(z) \, dz = \int_\gamma F'(z) \, dz$$

$$= \int_0^{2\pi} F'(\gamma(\theta)) \cdot \gamma'(\theta) \, d\theta$$

$$= F(\gamma(\theta)) \Big]_0^{2\pi}$$

$$= F(\gamma(2\pi)) - F(\gamma(0))$$

$$= F(R + z_0) - F(R + z_0)$$

$$= 0 \qquad \square$$