2025-10-21 LECTURE 7 SAUL SCHLEIMER MA3B8

SUPPOSE U A DOMAIN. SUPPOSE $T = T(p,q,r) \subset U$ A TRIANGLE.

LAST TIME $\int_{\partial T} (a+bz)\,dz = 0$.

IN FACT: FOR ANY $P \in \mathbb{C}[z]$, $\int_{\partial T} P\,dz = 0$

THUS, FOR ANY ANALYTIC

$f: U \to \mathbb{C}$ [HAS POWER SERIES], $\int_{\partial T} f\,dz = 0$.

"PROOF": $f_n \to f$ UNIFORMLY ON T AND $\int_{\partial T} f_n\,dz = 0$. $\square$

HERE IS A (MUCH!) MORE GENERAL APPROACH.

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(1) GOURSAT'S LEMMA: SUPPOSE $f: U \to \mathbb{C}$ IS HOLOMORPHIC

SUPPOSE $T = T(p,q,r) \subset U$ IS A TRIANGLE.

THEN $\int_{\partial T} f\,dz = 0$.

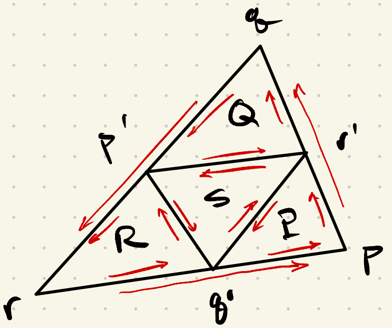
WE SAY "f INTEGRATES TO ZERO ABOUT TRIANGLES".

PROOF: FOR A CONTOUR $\gamma: [a,b] \to U$ DEFINE

$I(\gamma) = \int_\gamma f\,dz$.

CLAIM: $I(\partial T) = I(\partial P) + I(\partial Q) + I(\partial R) + I(\partial S)$.

PROOF: LINEARITY WITH RESPECT TO THE DOMAIN. $\square$



EXERCISE: GIVE A DETAILED PROOF, WITH PIECEWISE LINEAR PARAMETRISATIONS OF THE CONTOURS.

SO: BY Δ · INEQUALITY

$$|I(\partial T)| \leq |I(\partial P)| + |I(\partial Q)| + |I(\partial R)| + |I(\partial S)|$$

SET $T_0 = T$. SET $T_1 = P, Q, R,$ or S:

WHICHEVER GIVES

$$|I(\partial T_1)| = \text{MAX}\{|I(\partial P)|, |I(\partial Q)|, |I(\partial R)|, |I(\partial S)|\}$$

<u>THUS</u>:
$$\left.\begin{aligned} |I(\partial T_1)| &\geq \tfrac{1}{4} \cdot |I(\partial T_0)| \\ L(\partial T_1) &= \tfrac{1}{2} \cdot L(\partial T_0) \\ D(\partial T_1) &= \tfrac{1}{2} \cdot D(\partial T_0) \end{aligned}\right\} \text{ SEE FIGURE}$$

<u>REPEAT</u>: FIND SEQUENCE of TRIANGLES $(T_n)_n$

BY MIDPOINT SUBDIVISION WITH $T_{n+1} \subset T_n$

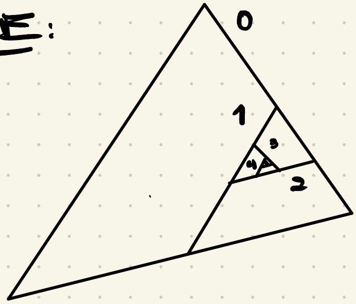
AND
$$\begin{aligned} |I(\partial T_{n+1})| &\geq \tfrac{1}{4} \cdot |I(\partial T_n)| \\ L(\partial T_{n+1}) &= \tfrac{1}{2} \cdot L(\partial T_n) \\ D(\partial T_{n+1}) &= \tfrac{1}{2} \cdot D(\partial T_n) \end{aligned}$$

<u>INDUCT</u>:
$$\begin{aligned} |I(\partial T_n)| &\geq \tfrac{1}{4^n} \cdot |I(\partial T_0)| \\ L(\partial T_n) &= \tfrac{1}{2^n} \cdot L(\partial T_0) \\ D(\partial T_n) &= \tfrac{1}{2^n} \cdot D(\partial T_0) \end{aligned}$$

NOTE THAT T_n ARE CLOSED, NESTED, AND

$D(T_n) \longrightarrow 0$. SO $\bigcap_n T_n = \{z_0\}$ IS A POINT

PICTURE:



RECALL (LEMMA) THAT:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \rho(z)(z - z_0)$$

ρ: $\begin{cases} \text{CONTINUOUS,} \\ \rho(z_0) = 0 \end{cases}$

FIX $\varepsilon > 0$. SO THERE IS A $\delta > 0$ SO THAT FOR ANY $w \in B(z_0, \delta)$ WE HAVE $|\rho(w)| < \varepsilon$.

FIX n SO THAT $T_n \subset B(z_0, \delta)$. NOW COMPUTE

$$I(\partial T_n) = \int_{\partial T_n} f \, dz$$

$$= \int_{\partial T_n} [f(z_0) + f'(z_0)(z - z_0) + \rho(z)(z - z_0)] \, dz$$

0 BY EXERCISE

$$= \cancel{\int_{\partial T_n} \big(f(z_0) + f'(z_0)(z - z_0)\big) dz} + \int_{\partial T_n} \rho(z)(z - z_0) \, dz$$

NOW: $\max\{|\rho(z)|\} \leq \varepsilon$ AND $\max\{|z - z_0|\} \leq D(\partial T_n)$

BY THE ML INEQUALITY

$$|I(\partial T_n)| = \left| \int_{\partial T_n} \rho(z)(z - z_0) \, dz \right| \leq \varepsilon \cdot D(\partial T_n) \cdot L(\partial T_n)$$

$$= \varepsilon \, \frac{D(\partial T_0)}{2^n} \cdot \frac{L(\partial T_0)}{2^n}$$

So: $\frac{1}{4^n}|I(\partial T_0)| \leq |I(\partial T_n)| \leq \frac{\varepsilon}{4^n} D(\partial T_0) \cdot L(\partial T_0)$

So: $|I(\partial T_0)| \leq \varepsilon \cdot D(\partial T_0) \cdot L(\partial T_0)$, FOR ALL $\varepsilon > 0$.

So: $|I(\partial T_0)| = 0$, SO: $I(\partial T_0) = 0$ $\square$

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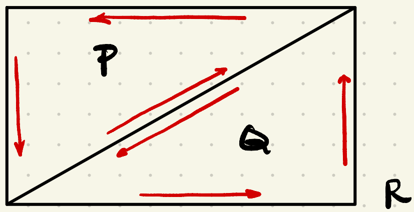
② EXAMPLE: HOLOMORPHIC FUNCTIONS INTEGRATE TO ZERO ABOUT RECTANGLES.

PROOF BY PICTURE: CUT THE RECTANGLE R INTO TRIANGLES P, Q

THE INTEGRALS ALONG THE DIAGONALS CANCEL SO



$I(R) = I(P) + I(Q) = 0 + 0.$ $\square$

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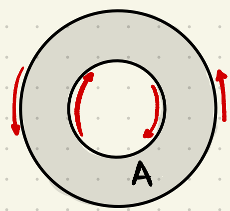
③ ANNULI: DEFINE

$$A = A(z_0; r, R) = \{ z \in \mathbb{C} \mid r < |z - z_0| < R \}$$

TO BE THE ANNULUS CENTRED AT z_0 WITH INNER RADIUS r AND OUTER RADIUS R.

QUESTION: ORIENTING ∂A AS SHOWN AND TAKING f HOLOMORPHIC, DO WE HAVE $\int_{\partial A} f \, dz = 0$?



WE CAN DECOMPOSE A INTO "CURVY TRIANGLES"....