2025-10-17: LECTURE 6 SAUL SCHLEIMER MA388

① PRIMITIVES:

SUPPOSE $f: U \to \mathbb{C}$ IS HOLOMORPHIC. LET $g = f'$. WE CALL f A PRIMITIVE (ANTIDERIVATIVE) of g.

LEMMA: SUPPOSE $f: U \to \mathbb{C}$ HOLOMORPHIC. SUPPOSE $\gamma: [a,b] \to U$ IS A CLOSED CONTOUR. THEN $\int_\gamma f'\,dz = 0$.

THAT IS: "FUNCTIONS WITH PRIMITIVES INTEGRATE TO ZERO ABOUT CLOSED CONTOURS".

PROOF.

$$\int_\gamma f'\,dz = \int_a^b f'(\gamma(t)) \cdot \gamma'(t)\,dt \qquad \text{DEF}$$

$$= f(\gamma(t))\Big]_a^b \qquad \text{FUND THM of CALCULUS}$$

$$= f(\gamma(b)) - f(\gamma(a)) \qquad \text{DEF}$$

$$= f(\gamma(a)) - f(\gamma(a)) \qquad \gamma \text{ CLOSED}$$

$$= 0 \qquad \square$$

SEE LECTURE CAPTURE

COROLLARY: POLYNOMIALS INTEGRATE TO ZERO ABOUT CLOSED CONTOURS.

PROOF: POLYNOMIALS HAVE PRIMITIVES. $\square$

CONTRAST THIS WITH $\int_C \frac{dz}{z} = 2\pi i \neq 0$.

(1) **SEGMENTS:** SUPPOSE $p,q \in \mathbb{C}$. $p+t(q-p)$ ~60

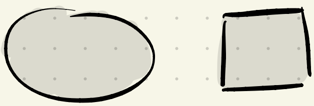
DEFINE $[p,q] = \{(1-t)p + tq \mid t \in [0,1]\}$

(*) INTEGRATE $\int_{[p,q]} dz$, $\int_{[p,q]} z\,dz$.

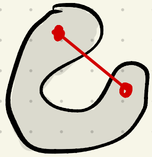
(1) **CONVEXITY**

DEFINE: SAY $K \subset \mathbb{C}$ IS **CONVEX** IF, FOR ALL $p,q \in K$, WE HAVE $[p,q] \subset K$.

EXAMPLE:



NONEXAMPLE



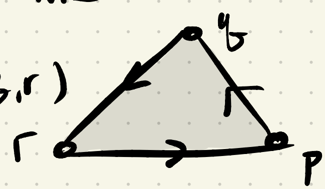
EXERCISE: PROVE $B(z_0;R)$ IS CONVEX.

DEFINE: FOR $S \subset \mathbb{C}$, $\mathrm{HULL}(S) = \bigcap \left\{ K \;\middle|\; \begin{array}{l} K \text{ CONVEX} \\ S \subset K \end{array} \right\}$

EXERCISE: $\overline{B(0;1)}$ IS NOT THE CONVEX HULL OF ANY COUNTABLE SET. [BUT $\mathbb{C}$ IS!].

(2) **TRIANGLES** SUPPOSE $p,q,r \in \mathbb{C}$. DEFINE THE

TRIANGLE $T = T(p,q,r) = \mathrm{HULL}(p,q,r)$



DEFINE ∂T TO BE THE CONTOUR

$$[p,q] \cup [q,r] \cup [r,p]$$

CALL T **POSITIVELY ORIENTED** IF

i) T NOT DEGENERATE [p,q,r COLINEAR]

ii) p,q,r ARRANGED ANTI-CLOCKWISE

[NEG ORIENTED IF CLOCKWISE].

③ INTEGRALS ABOUT TRIANGLES

EXERCISES SUPPOSE $T = T(p,q,r)$ A TRIANGLE

(1) $\int_{\partial T} dz = 0$ (2) $\int_{\partial T} z\, dz = 0.$

(3) $\int_{\partial T} f(z)\,dz = 0$ FOR $f \in \mathbb{C}[z]$ ANY POLYNOMIAL.

[RMK: OR $f = \sum a_n z^n$ ANALYTIC]

④ SUBDIVISION:

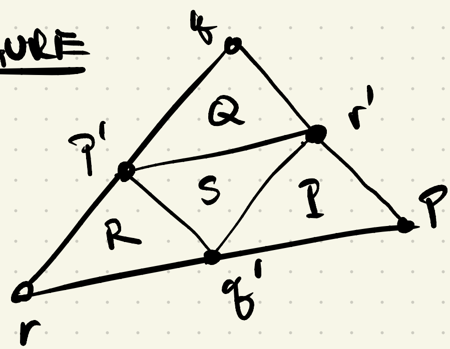
GIVEN $T = T(p,q,r)$ DEFINE THE MIDPOINTS

$$p' = \frac{q+r}{2} \quad q' = \frac{r+p}{2} \quad r' = \frac{p+q}{2}$$

AND $P = T(p, r', q')$, $Q = (q, p', r')$, $R = T(r, q', p')$

AND $S = T(p', q', r')$.

FIGURE



THIS IS THE MIDPOINT SUBDIVISION of T.

NOTE P, Q, R, S CONGRUENT TO EACH OTHER. ALSO ALL SIMILAR TO T (SCALE BY FACTOR of $1/2$).

LEMMA:

ARC LENGTH: $\frac{1}{2} L(\partial T) = L(\partial P) = L(\partial Q) = L(\partial R) = L(\partial S)$

DIAMETER: $\frac{1}{2} D(\partial T) = D(\partial P) = D(\partial Q) = D(\partial R) = D(\partial S)$

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