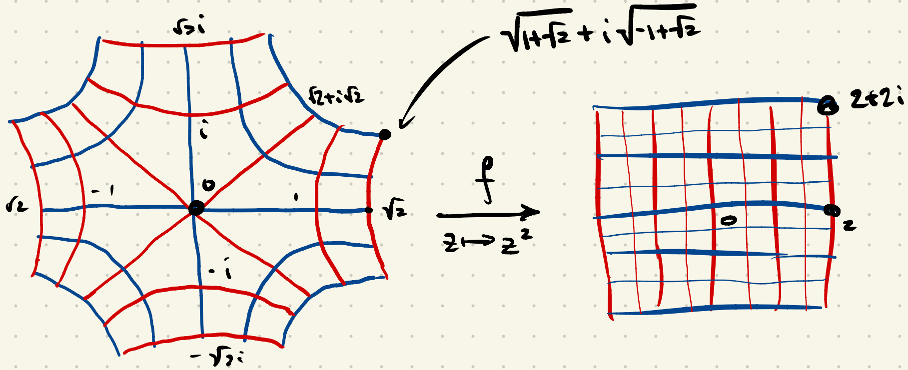
2025-10-15 LECTURE 5 MA3B8 SAUL SCHLEIMER ~70

EXERCISE: DRAW THE PREIMAGE OF THE GRID, UNDER $f(z) = z^2$



EXERCISE: CHECK THAT THESE ARE HYPERBOLAS.

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① CONTINUITY

LEMMA SUPPOSE $f: U \to \mathbb{C}$ IS HOLOMORPHIC.
THEN f IS CONTINUOUS (SO INTEGRABLE) $\square$

LEMMA: SUPPOSE $f: U \to \mathbb{C}$ IS HOLOMORPHIC. FIX $z_0 \in U$
DEFINE

$$p(z) = \begin{cases} 0, & \text{if } z = z_0 \\ \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0), & \text{if } z \neq z_0 \end{cases}$$

THEN: $p: U \to \mathbb{C}$ IS CONTINUOUS

PROOF SKETCH: USE EXERCISE IF $z \neq z_0$.
IF $z = z_0$ USE HOLOMORPHICITY. $\square$

② POWER SERIES: THESE GENERALISE POLYNOMIALS.

AS USUAL, for $(a_k) \subset \mathbb{C}$. DEFINE $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$.

AND $\frac{1}{R} = \limsup |a_n|^{1/n}$. [WITH $R = 0, \infty$ ALLOWED]

WE CALL R THE RADIUS of CONVERGENCE of THE SERIES.

DEFINE $f_N(z) = \sum_{k=0}^{N} a_k (z - z_0)^k$. THESE ARE THE PARTIAL SUMS of f.

THESE HAVE ALL of THE SAME PROPERTIES AS IN THE REAL CASE; THERE IS A LIST IN THE NOTES.

EXAMPLE: $f(z) = \frac{1}{1+z^2}$ HAS SERIES EXPANSION

$$\sum (-1)^k z^{2k} = 1 - z^2 + z^4 - z^6 + z^8 - \cdots$$

THE RADIUS of CONVERGENCE IS ONE. NOTE THAT f "BLOWS UP" AT $z = i$ AND $|i - 0| = 1$.

EXERCISE: GIVE SERIES AND RADIUS FOR

$$f(z) = \frac{1}{1+z+z^2}.$$ HINT: $f(z) = \frac{1-z}{1-z^3}$

③ COMPLEX EXPONENTIAL: $\mathrm{EXP}(z) = \sum \frac{z^k}{k!}$

$\cos(z) = \sum (-1)^k \frac{z^{2k}}{(2k)!}$, $\sin(z) = \sum (-1)^k \frac{z^{2k+1}}{(2k+1)!}$

<u>EXERCISES</u>:
1. $\exp'(z) = \exp(z)$
2. $\exp(z+w) = \exp(z)\exp(w)$.
3. $\exp(iz) = \cos(z) + i\sin(z)$
4. $\exp(z) = \exp(w)$ IFF $z - w \in 2\pi i\,\mathbb{Z}$.
5. $\exp(\mathbb{C}) = \mathbb{C}^\times$.

<u>EXAMPLE</u>: $F(z) = \exp(z)\exp(w-z)$

$F'(z) = \exp(z)\exp(w-z) - \exp(z)\exp(w-z) = 0$

$F(0) = \exp(w)$. THUS $\exp(z)\cdot\exp(w-z) = \exp(w)$. NOW SUB. $\square$

<u>DEFINE</u>: $\mathrm{LOG}(w) = \{ z \in \mathbb{C} \mid \exp(z) = w \}$.

<u>LEMMA</u>: $\mathrm{LOG}(w) = \log(|w|) + i\,\mathrm{ARG}(w)$.

<u>PROOF</u>: $\exp(\log(|w|) + i\,\mathrm{ARG}(w)) =$

$= \exp(\log(|w|))\exp(i\,\mathrm{ARG}(w))$

$= |w|[\cos(\mathrm{ARG}(w)) + i\sin(\mathrm{ARG}(w))]$

$= |w|\, e^{i\,\mathrm{ARG}(w)} = w.$ $\square$

DEDUCE $\mathrm{LOG}(w)$ IS A "CONTINUOUS" MULTI-VALUED FUNCTION. BUT WE MUST PROVE LOG IS <u>HOLOMORPHIC</u>.