① WINDING NUMBERS

LEMMA: SUPPOSE $\gamma:[a,b]\longrightarrow \mathbb{C}^\times$ IS A CLOSED CONTOUR. THEN $V(\gamma)\in 2\pi\mathbb{Z}$.

PROOF: LAST TIME WE FOUND $\theta:[a,b]\longrightarrow\mathbb{R}$ SO THAT θ IS CONTINUOUS AND $\gamma(t)=|\gamma(t)|\cdot e^{i\theta(t)}$ for ALL t.

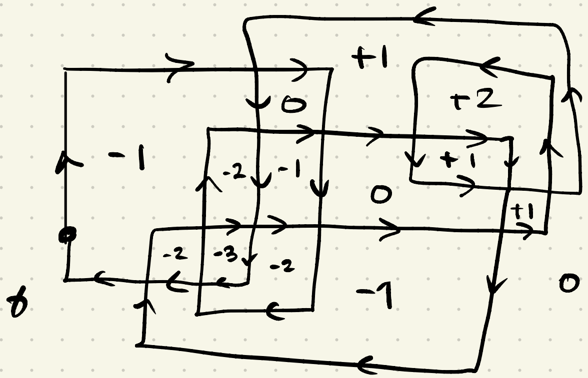
So $|\gamma(b)|e^{i\theta(b)}=|\gamma(a)|e^{i\theta(a)}$

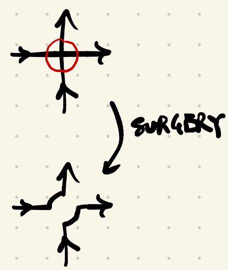
So $e^{i(\theta(b)-\theta(a))}=1$ So $V(\gamma)=\theta(b)-\theta(a)\in 2\pi\mathbb{Z}$. $\square$

DEFINE: FOR $z_0\notin\gamma([a,b])$

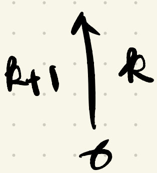
$$\mathrm{WIND}(\gamma,z_0)=\frac{V(\gamma-z_0)}{2\pi}$$

PICTURE:

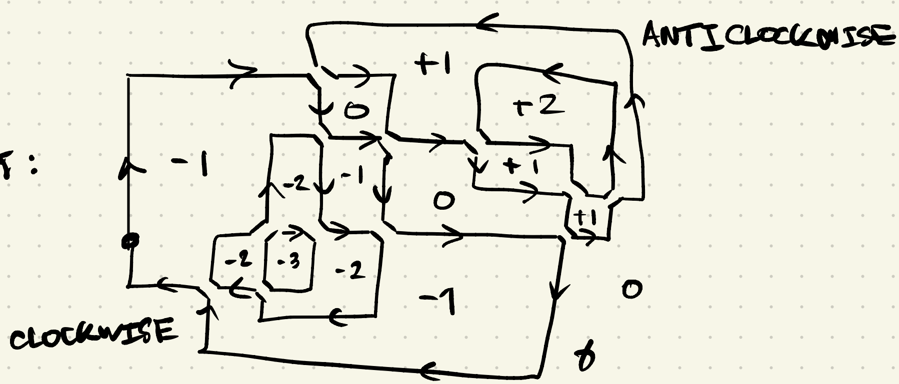




HERE IS THE RULE:



THERE IS ANOTHER WAY:



② HOLOMORPHIC FUNCTIONS

DEFINITION: SUPPOSE $U \subset \mathbb{C}$ IS A DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ IS A FUNCTION. SUPPOSE $z_0 \in U$. WE SAY f IS HOLOMORPHIC AT z_0 IF

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{EXISTS.}$$

[THAT IS: THERE IS SOME $c \in \mathbb{C}$ SO THAT FOR ALL $\varepsilon > 0$ THERE IS SOME $\delta > 0$ SO THAT IF $0 < |z - z_0| < \delta$ THEN $\left| \frac{f(z)-f(z_0)}{z - z_0} - c \right| < \varepsilon$]

IF f IS HOLOMORPHIC AT $z_0 \in U$ FOR ALL $z_0 \in U$ THEN SAY f IS HOLOMORPHIC IN U.

WE WRITE $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z - z_0}$. THUS $f': U \to \mathbb{C}$ IS AGAIN A FUNCTION. WE DO NOT ASSUME FURTHER REGULARITY OF f' [CONTINUITY, ETC]

EXAMPLES: ① $f(z) = z$ IS HOLOMORPHIC.

$f'(z) = 1$.

② POLYNOMIALS ARE HOLOMORPHIC.

③ $f(z) = \frac{1}{z}$ IS HOLOMORPHIC IN $\mathbb{C}^{\times}$.

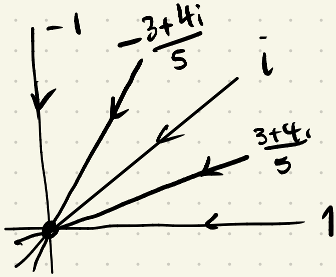
④ IF $f, g: U \to \mathbb{C}$ ARE HOLOMORPHIC SO ARE $f + g$ AND $f \cdot g$. AND ALSO f/g, IF $g(z_0) \neq 0$ FOR ALL $z_0 \in U$

NON-EXAMPLES : $f(z) = \mathrm{REAL}(z)$

$f(z) = \mathrm{IMAG}(z)$

$f(z) = \bar{z}$

WE COMPUTE:



FOR $f(z) = \bar{z}$ THE QUOTIENT IS CONSTANT ALONG STRAIGHT LINES BUT THE QUOTIENTS DO NOT AGREE

EXERCISE: TO THE SAME FOR $f(z) = \mathrm{REAL}(z)$

EXERCISE

If $f: U \to X$, $g: V \to W$ HOLOMORPHIC THEN SO IS $f \circ g: U \to W$ AND $(f \circ g)'(z_0) = f'(g(z_0)) \cdot g'(z_0)$

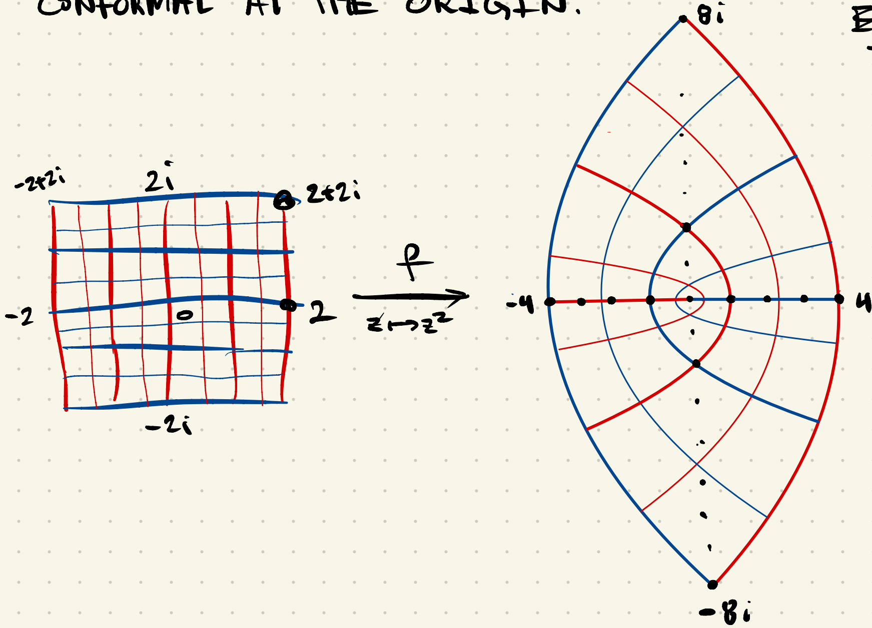
③ REAL DERIVATIVE

LEMMA: SUPPOSE $f: U \to \mathbb{C}$ IS HOLOMORPHIC. THEN f IS DIFFENTIABLE AS A REAL FCN of TWO VARIABLES. ALSO
$$D_{z_0} f = r \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$
WHERE $r = |f'(z_0)|$

AND $\theta \in \mathrm{ARG}(f'(z_0))$.

THAT IS: IF $f'(z_0) \neq 0$ THEN $D_{z_0}f$ IS A HOMOTHETY; SCALE PLUS ROTATION. IN PARTICULAR $D_{z_0}f$ PRESERVES SIGNED ANGLES. THIS PROPERTY IS CALLED CONFORMALITY.

EXAMPLE: $f(z)=z^2$. NOTE $f'(0)=0$ SO f IS NOT CONFORMAL AT THE ORIGIN.



EXERCISE
THE GRIDLINES ARE SENT TO PARABOLAS.

REMARK: $f(z)=z^2$ IS NOT CONFORMAL AT THE ORIGIN BECAUSE $f'(0)=0$. INSTEAD IT IS "ANGLE DOUBLING" AT $z_0=0$.

EXERCISE: DRAW THE "SAME PICTURE" FOR $f(z)=z^3$: