2025-10-10 LECTURE 3 MA3B8 SAUL SCHLEIMER

QUESTION ABOUT TYPED NOTES:

$\square$ MEANS END of PROOF / SOLUTION.

$\diamond$ " " " EXAMPLE / DEFINITION.

THERE IS AN INDEX of SYMBOLS; LET ME KNOW AND I WILL ADD TO IT.

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(1) ARC-LENGTH. WE DEFINE THE ELEMENT of ARC-LENGTH TO BE

$$ds = \sqrt{dx^2 + dy^2} = |dz|$$

SUPPOSE $\gamma : [a,b] \longrightarrow U$ IS A CONTOUR. THEN

WRITE $\gamma(t) = (x(t), y(t)) = x(t) + i y(t)$

AND $\gamma'(t) = (x'(t), y'(t)) = x'(t) + i y'(t)$

SO: $|\gamma'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$

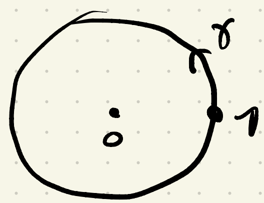
DEFINE

$$\int_\gamma ds = \int_\gamma \sqrt{dx^2 + dy^2} = \int_a^b |\gamma'(t)|\, dt$$

EXAMPLE: $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$, $\gamma(\theta) = e^{i\theta} = \cos(\theta) + i \sin(\theta)$

So: $\gamma'(\theta) = i e^{i\theta}$ AND $|\gamma'(\theta)| = 1$.

SO $\int_\gamma ds = \int_0^{2\pi} 1 \cdot d\theta = 2\pi.$



**LEMMA**: ARC-LENGTH IS INSENSITIVE TO REPARAMETRISATION: IF $\gamma : [a,b] \longrightarrow \mathbb{C}$ AND $\delta : [c,d] \longrightarrow [a,b]$ IS PIECEWISE $C^1$ AND BIJECTIVE THEN $\int_\gamma ds = \int_{\gamma \circ \delta} ds$. $\square$

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② **THE ML-INEQUALITY**: IS <u>VERY</u> <u>VERY</u> HANDY.

**LEMMA**: SUPPOSE $U \subset \mathbb{C}$ A DOMAIN. SUPPOSE $f : U \to \mathbb{C}$ IS CONTINUOUS. SUPPOSE $\gamma : [a,b] \longrightarrow U$ IS A CONTOUR. SUPPOSE $M = \max\{f(\gamma(t)) \mid t \in [a,b]\}$ AND $L = \int_\gamma ds$

THEN $\left| \int_\gamma f \, dz \right| \leq M \cdot L$

**PROOF**: $\left| \int_\gamma f \, dz \right| \leq \int_\gamma |f| \, |dz| \leq \int_\gamma M \cdot ds = M \cdot L$ $\square$

**EXERCISE**: EXPAND THIS (USING DEFINITIONS) TO GET A "NONABBREVIATED" PROOF.

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③ **ELEMENT of ARGUMENT**:

WE DEFINE THIS TO BE: $d\theta = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$

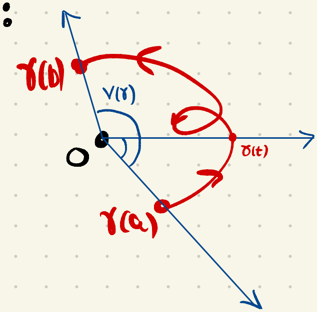
SUPPOSE $\gamma : [a,b] \longrightarrow \mathbb{C}^\times$ IS A CONTOUR.

TAKE $\gamma(t) = x(t) + i y(t)$. DEFINE

$$V(\gamma) = \int_\gamma \frac{x\,dy - y\,dx}{x^2 + y^2} = \int_a^b \frac{x(t) y'(t) - y(t) x'(t)}{|\gamma(t)|^2} \, dt$$

THIS IS THE VARIATION of ARGUMENT.
IT IS INSENSITIVE TO (ORIENTATION PRESERVING) REPARAMETRISATION.

PICTURE:




PROPOSITION: FIX $\gamma:[a,b]\to\mathbb{C}^\times$ A CONTOUR. FIX $\theta_a \in \mathrm{ARG}(\gamma(a))$. THEN THERE IS A UNIQUE CONTINUOUS FUNCTION $\theta:[a,b]\longrightarrow\mathbb{R}$ SO THAT

(i) $\theta(a)=\theta_a$

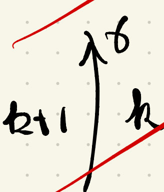
(ii) $\theta(t)\in \mathrm{ARG}(\gamma(t))$ FOR ALL $t\in[a,b]$.

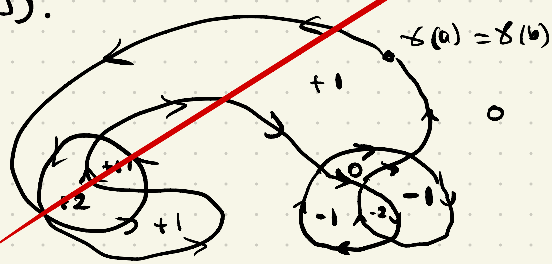
PROOF: DEFINE $\theta(t)=\theta_a+V(\gamma|[a,t])$. THIS IS CONTINUOUS BECAUSE IT IS AN INTEGRAL. [NOTE: $x'(t)$ AND $y'(t)$ ARE PIECEWISE CONTINUOUS].
DEFINE $r(t)=|\gamma(t)|$. DEFINE $h(t)=\dfrac{r(t)e^{i\theta(t)}}{\gamma(t)}$.

EXERCISE $h'(t)=0$.

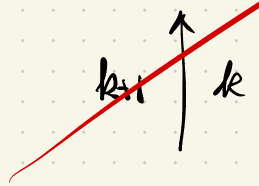
ALSO: $h(a)=1$. SO $\gamma(t)=r(t)e^{i\theta(t)}$ $\square$.

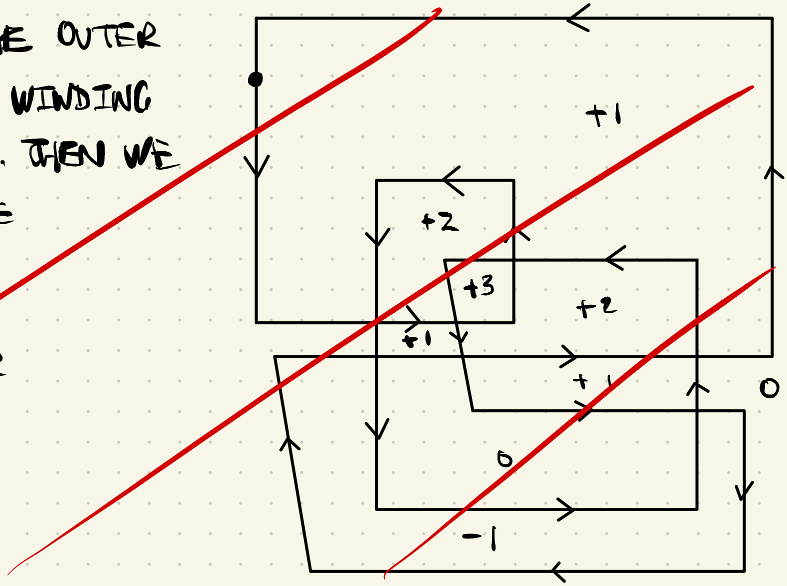
EXERCISE: SUPPOSE $\gamma$ IS A CLOSED CONTOUR IN $\mathbb{C}^\times$. THEN $V(\gamma)=2\pi k$ FOR SOME $k\in\mathbb{Z}$. WE CALL $k$ THEN WINDING NUMBER of $\gamma$ ABOUT ZERO.
WE WRITE $\mathrm{WIND}(\gamma, z_0)$ FOR THE WINDING of $\gamma$ ABOUT $z_0$ FOR $z_0\notin\gamma([a,b])$.



POINTS IN THE OUTER REGION HAVE WINDING NUMBER ZERO. THEN WE HAVE THE RULE

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④ <u>ELEMENT of RADIUS</u>:

RECALL THAT $\qquad r^2 = x^2 + y^2.$

THUS $\qquad 2r\,dr = 2x\,dx + 2y\,dy$

SO $$dr = \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$$

WE CALL THIS THE <u>ELEMENT of RADIUS</u>

<u>EXERCISE</u>: $\displaystyle\int_\gamma dr = \int_\gamma \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = |\gamma(b)| - |\gamma(a)|.$

FOR CONTOUR $\gamma : [a,b] \longrightarrow \mathbb{C}^\times$.

⑤ <u>ELEMENT of THE LOGARITHM</u>

<u>DEFINE</u>: $\displaystyle\frac{dr}{r} = \frac{x\,dx + y\,dy}{x^2+y^2}$ TO BE THE ELEMENT

of $\log(r)$ [THE LOG OF THE RADIUS].
WE SIMILARLY DEFINE THE ELEMENT OF LOG(z) TO BE $dz/z$. HERE IS A FINAL CONSISTENCY CHECK [I.E. "WAY TO REMEMBER EVERYTHING"]

SUPPOSE $z = re^{i\theta}$. So $dz = e^{i\theta}dr + rie^{i\theta}d\theta$.

$$\text{so} \quad \frac{dz}{z} = \frac{dr}{r} + i\,d\theta$$

$$= \frac{x\,dx + y\,dy + i(x\,dy - y\,dx)}{x^2+y^2}$$

$$= \frac{x\,dx - iy\,dx + y\,dy + ix\,dy}{x^2+y^2}$$

$$= \frac{(x-iy)\,dx + (y+ix)\,dy}{(x-iy)(x+iy)}$$

$$= \frac{\cancel{(x-iy)}(dx+i\,dy)}{\cancel{(x-iy)}(x+iy)}$$

$$= \frac{dz}{z}$$