2025-10-08 LECTURE 2 MA388 SAUL SCHLEIMER ~85 STUDENTS

① DOMAINS ARE NOT ALL BAD.

LEMMA: SUPPOSE $U \subset \mathbb{C}$ IS A DOMAIN. THEN U IS PATH-CONN.

PROOF: WE "CRAWL". FIX $p \in U$.

DEFINE $U_p = \left\{ q \in U \;\middle|\; \begin{array}{l} \text{THERE IS SOME} \\ \text{PATH FROM } p \text{ TO} \\ q \text{ IN } U \end{array} \right\}$

NOTE U_p NONEMPTY, PATH-CONN, AND OPEN.
FIX ANY $q \in U - U_p$. SINCE U OPEN FIND $\varepsilon > 0$
SO THAT $B(q;\varepsilon) \subset U$. IF $B(q;\varepsilon) \cap U_p \neq \emptyset$
HAVE CONTRADICTION. SO $B(q;\varepsilon) \subset U - U_p$. SO
$U - U_p$ OPEN. SO $U - U_p = \emptyset$. □.

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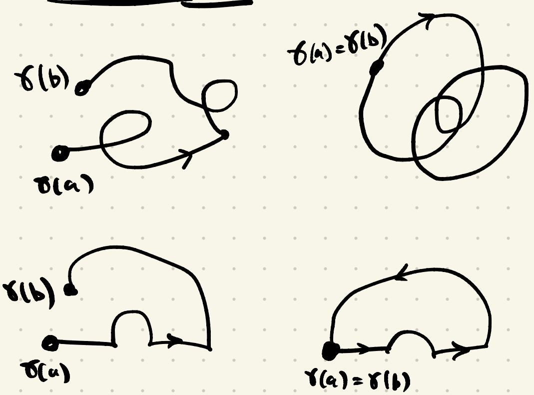
② CONTOURS: A CONTOUR IN U IS A CONTINUOUS
FUNCTION $\gamma : [a,b] \longrightarrow U$ AND A PARTITION
$(a = t_0 < t_1 < t_2 < \cdots < t_N = b)$ SO THAT $\alpha|[t_j, t_{j+1}]$
IS C^1 (CONT. DIFF) WITH NON-VANISHING DERIV.

PICTURES



DEF: THE CONTOUR IS CLOSED IF $\gamma(a) = \gamma(b)$.

DEF THE CONTOUR IS SIMPLE IF

$\gamma(s) = \gamma(t)$ IMPLIES

$s = t$ OR $\{s,t\} = \{a,b\}$.

IF $\gamma : [a,b] \longrightarrow \mathbb{C}$ IS A CONTOUR, WE WILL IDENTIFY $\gamma'(t) = (\gamma_x'(t), \gamma_y'(t))$ WITH $\gamma_x'(t) + i\gamma_y'(t)$

3 CONTOUR INTEGRALS:

DEF: SUPPOSE $U \subset \mathbb{C}$ IS A DOMAIN. SUPPOSE $f : U \to \mathbb{C}$ IS CONTINUOUS. SUPPOSE $\gamma : [a,b] \longrightarrow U$ IS A CONTOUR.

DEFINE
$$\int_\gamma f \, dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t) \, dt.$$

IF γ HAS BREAKPTS / CORNERS THEN WRITE RHS AS A SUM OVER THE PIECES. [ALWAYS CHECK CONVERGENCE!]

EXAMPLE: THE ONE TRUE EXAMPLE.

$U = \mathbb{C}^\times = \mathbb{C} - \{0\}$. $f(z) = 1/z$, $\gamma : [0, 2\pi] \longrightarrow U$
$\gamma(\theta) = e^{i\theta}$.

THEN:
$$\int_\gamma f \, dz = \int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{i\cancel{e^{i\theta}}}{\cancel{e^{i\theta}}} \, d\theta = 2\pi i$$

EXAMPLES

$$\int_c \frac{dz}{2z-1} = ?$$

$$\int_c \frac{dz}{z-1} \text{ DNE}$$

EXERCISE: FOR $n \in \mathbb{Z}$
$$\int_\gamma \frac{dz}{z^n} = \begin{cases} 2\pi i, & \text{if } n = 1 \\ 0, & \text{if } n \neq 1 \end{cases}$$

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4 LEMMA: SUPPOSE U IS A DOMAIN. SUPPOSE $p, q \in U$. SUPPOSE $f, g : U \to \mathbb{C}$ ARE CONT. SUPPOSE $\gamma : [a,b] \to U$ IS A CONTOUR. THEN
$$\int_\gamma (pf + qg) \, dz = p \cdot \int_\gamma f \, dz + q \cdot \int_\gamma g \, dz. \qquad \square$$

"LINEARITY IN THE INTEGRAND".

LEMMA: WITH U, f, γ AS ABOVE. SUPPOSE $c \in [a,b]$ SETTING $\gamma_a = \gamma|[a,c]$, $\gamma_b = \gamma|[c,b]$ WE HAVE

$$\int_\gamma f\,dz = \int_{\gamma_a} f\,dz + \int_{\gamma_b} f\,dz. \qquad \square$$

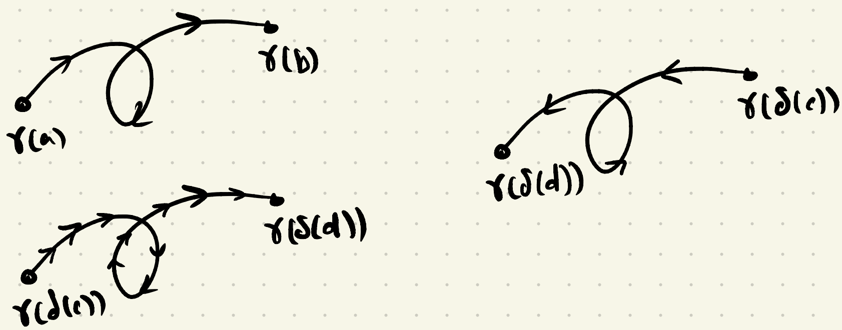
"LINEARITY IN THE DOMAIN of INTEGRATION"

DEF: SUPPOSE $\gamma : [a,b] \longrightarrow U$ IS A CONTOUR. SUPPOSE $\delta : [c,d] \longrightarrow [a,b]$ IS PIECEWISE C^1. WITH NON-VANISHING DERIVATIVE.

IF $\delta(c) = a$, $\delta(d) = b$ THEN $\int_{\gamma\circ\delta} f\,dz = \int_\gamma f \cdot dz$

If $\delta(c) = b$, $\delta(d) = a$ THEN $\int_{\gamma\circ\delta} f\,dz = -\int_\gamma f\,dz$.

PICTURE



SAME ORIENTION OPPOSITE ORIENTATION

PROOF: CHAIN RULE $\square$