2025-10-07 MA388 LECTURE 1 SAVLSCHEIMER

ADMIN: SCHEDULE, NOTES, EXAMPLE SHEETS ON LINE.
100% EXAM (SORRY!)
TWO FORUMS FOR ANNOUNCEMENTS/QUESTIONS.

OUTLINE: HOLOMORPHIC FUNCTIONS
CAUCHY'S THY of CONTOUR INTEGRALS
WEIERSTRASS'S THY of ANALYTIC FCNS
RIEMANN'S THY of CONFORMAL EQUIV.

PLEASE: ASK QUESTIONS!

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① NOTATION:

$\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ USUAL MEANINGS.

FOR $z = x + iy \in \mathbb{C}$ WRITE: $\mathrm{REAL}(z) = x$. $\mathrm{IMAG}(z) = y$

$\bar{z} = x - iy$, $|z| = \sqrt{x^2+y^2}$ SO $z\bar{z} = |z|^2$, $|zw| = |z||w|$.
$|z+w| \le |z| + |w| \dots$

DEF: $B(z_0; r) = \{ z \in \mathbb{C} \mid |z - z_0| < r \}$ OPEN BALL (DISK)

$C(z_0, r) = \{ z \in \mathbb{C} \mid |z - z_0| = r \}$ CIRCLE

$\mathbb{D} = B(0; 1)$ UNIT DISK, $\mathbb{D}^\times = \mathbb{D} - \{0\}$

$\mathbb{H} = \{ z \in \mathbb{C} \mid \mathrm{IMAG}(z) > 0 \}$, $\mathbb{C}^\times = \mathbb{C} - \{0\}$.

② POLAR COORDS: FIX $z = x + iy \neq 0$

DEFINE $c = \dfrac{x}{\sqrt{x^2+y^2}}$ $s = \dfrac{y}{\sqrt{x^2+y^2}}$.

SO $c^2 + s^2 = 1$. CALL $\theta \in \mathbb{R}$ AN ARGUMENT of z if

$C = \cos(\theta)$, $S = \sin(\theta)$.

DEF $\operatorname{ARG}(z) = \{\theta \in \mathbb{R} \mid \theta \text{ ARGUMENT of } z\}$.

NOTE: $\operatorname{ARG}(0)$ IS NOT DEFINED.

EXERCISE: FOR $z, w \neq 0$

(I) $\operatorname{ARG}(z) \neq \emptyset$,

(II) $\operatorname{ARG}(z) = \{\theta + 2\pi k \mid k \in \mathbb{Z}\}$ for ANY ARGUMENT θ.

(III) $\operatorname{ARG}(zw) = \operatorname{ARG}(z) + \operatorname{ARG}(w)$.

CALL ARG A "MULTI-VALUED" FUNCTION.

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③ DEF: SUPPOSE $U \subset \mathbb{C}$ IS NONEMPTY, CONNECTED, AND OPEN. WE CALL U A DOMAIN.

DEF: A BRANCH of ARG IS

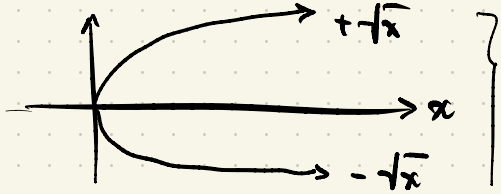
(i) A DOMAIN $U \subset \mathbb{C}^\times = \mathbb{C} - \{0\}$ AND

(ii) A CONTINUOUS FUNCTION $g: U \to \mathbb{R}$ SO THAT $g(z) \in \operatorname{ARG}(z)$ for ALL $z \in U$.

REMARK: UNDERSTANDING BRANCHES IS ONE of THE NEW FEATURES of COMPLEX ANALYSIS.

EXAMPLE: IN REAL ANALYSIS $\sqrt{x}$ IS UNDEF FOR $x < 0$, AND HAS TWO BRANCHES for $x > 0$.

PICTURE:

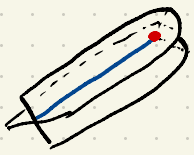


GRAPHS of $\pm\sqrt{x}$.

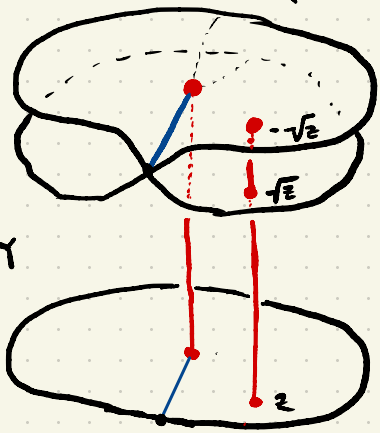
INVERSE FUNCTIONS to $\mathbb{C}$-ANALYTIC FUNCTIONS ARE MORE ORNATE: THE CHOICE of DOMAIN VARIES WITH THE PROBLEM.

HERE IS AN ATTEMPT TO DRAW THE "GRAPH" of THE "FUNCTION" $z \longmapsto \sqrt{z}$

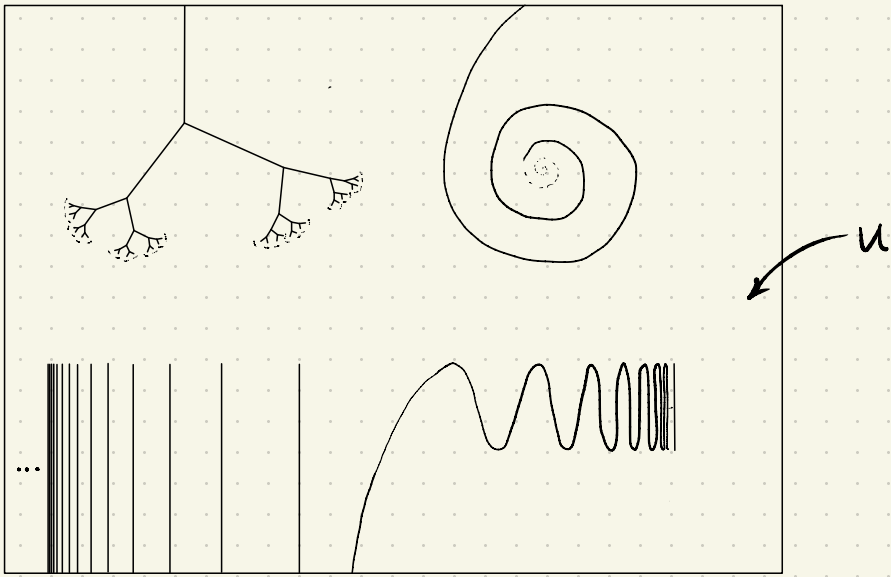
HERE THE ARC of INTERSECTIONS



DRAWN IN BLUE ARE AN ARTIFACT of THE LOW-DIMENSIONALITY of THE PICTURE.



④ DOMAINS : CAN BE "WILD"



HERE U IS THE REGION INSIDE THE RECTANGLE MINUS THE FOUR CLOSED SETS: DENDRITE (UPPER LEFT), SPIRAL (UPPER RIGHT), COMB (LOWER LEFT), AND TOPOLOGIST SINE CURVE (LOWER RIGHT).

NONETHELESS: THIS DOMAIN U IS BIHOLOMORPHIC TO THE UNIT DISK $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$!